Vocabulary: ***Accentuate the Negative***

Concept

Integers: The set of whole numbers and their **opposites.** (Opposites are also called **additive inverses.**)

Opposites: Numbers which are on opposite sides of zero on the number line, the same distance from zero. Numbers greater than zero are called **positives.** Numbers less than zero are called **negative numbers.**

Rational numbers: The set of integers and positive and negative fractions. (Or the set of numbers made by dividing any 2 integers, excepting division by zero.)

Ordering Rational Numbers: Since the set of whole numbers is symmetrically placed on the number line with regard to the opposites of all the whole numbers, the order of the positive numbers is exactly opposite to the order of the negative numbers.

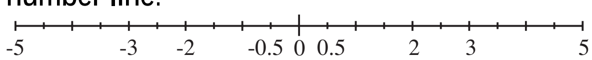
Notation: Positives can be written with or without a sign. Thus, 7 and +7 mean the same. Negative numbers must have a preceding sign. Thus, -7 means “negative 7.” The preceding sign may be raised or not. Thus, “negative 7” may be written -7 or $^{-}7$ or (-7). Usually the context distinguishes between “add” and “positive,” or “subtract” and “negative,” and a raised sign or parentheses will be used when numbers and operations might be confused. Thus, 4 + $^{-}3$ means “4 add negative 3,” or 4 **–** (-3) means “4 subtract negative 3.” (More on this later under “algorithms.”)

Example

2, 3, 6, 19, 1000 are all **whole numbers.** The **opposites** of these are -2, -3, -6, -19 and -1000. The result of adding a number and its opposite is zero. For example, 3 + (-3) = 0. Thus, 3 and -3 are **additive inverses.**

The set of **whole numbers** is {0, 1, 2, 3, 4, 5….}. Therefore, the set of **integers** is {…-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 ….}

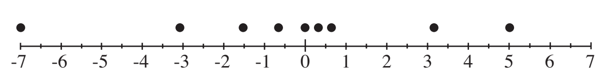
The **opposite** of 3 is -3. The opposite of -5 is 5. The opposite of -0.5 is 0.5 etc. Each number and its opposite are symmetrically placed on the **number line**:



Some examples of **Rational** numbers are:
$-\frac{3}{2}$, $-\frac{5}{7}$, $-\frac{14}{2}$ or -7, $\frac{0}{3}$ or 0, $3\frac{1}{7}$ or $\frac{22}{7}$, $\frac{10}{2}$ or 5, $\frac{1}{3}$, 0.6 or $\frac{6}{10}$, -3.1 or $-\frac{31}{10}$ etc.

If we are asked to **order the above list of rational numbers** we might think of the positives (0, $3\frac{1}{7}$, 5, $\frac{1}{3}$, 0.6) first:
$0 < \frac{1}{3} < 0.6 < 3\frac{1}{7} < 5$. (See *Bits and Pieces 1*)
Now, thinking of the negatives ($-\frac{3}{2}$, $-\frac{5}{7}$, -7, -3.1):
We know that 7> 3.1, so, taking advantage of the symmetry of opposites on the number line, we can say that -7 < -3.1. We can think of the “less than” sign as “further to the left on the number line.” Thus:
$-7 < -3.1 < -\frac{3}{2} < -\frac{5}{7}$.

Putting this together:
$-7 < -3.1 < -\frac{3}{2} < -\frac{5}{7} < 0 < \frac{1}{3} < 0.6 < 3\frac{1}{7} < 5$.

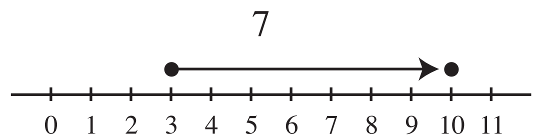


2 Models for Integers: Since different operations or contexts might be better understood in one model rather than another, students learn about 2 different models.
Number Line: This is useful for understanding order, addition, and some subtractions ("what is the difference?").
Chip Board: This is useful for understanding opposites, addition, and some subtractions ("If you take away XX what is left?").

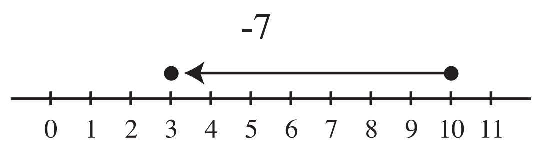
Number line for ordering: See above.

Number line for adding: See Algorithm for addition, below.

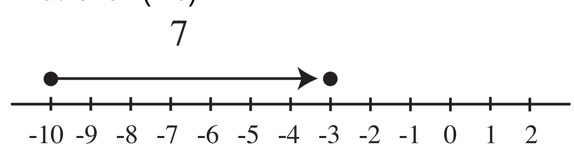
Number line for subtraction:
Find the difference between 3 and 10. This can be modeled as "starting at 3, what distance would it take to move to 10?" That is 10 – 3 = ?



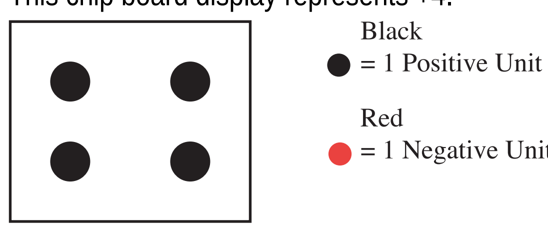
Find the difference between 10 and 3. "Starting at 10, what distance would it take to move to 3?" That is 3 – 10 = ?



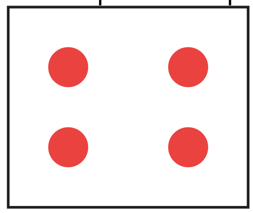
Find the difference between -10 and -3. "Starting at -10, what distance would it take to get to -3?" That is -3 –(-10) = ?



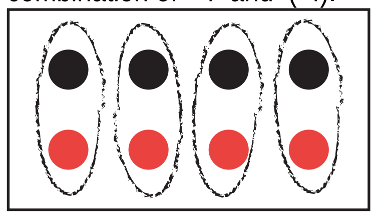
Chip Board for opposites:
This chip board display represents +4.



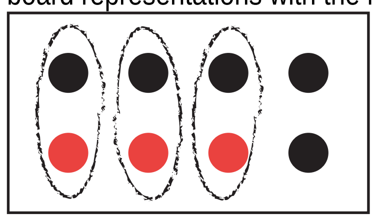
This chip board display represents -4.



This chip board display represents 0, a combination of +4 and (-4).



This chip board represents +2, a combination of +5 and (-3). Notice that there are lots of chip board representations with the net result of +2.

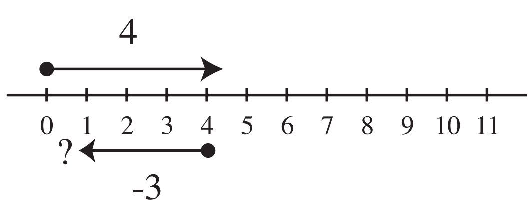


Addition Algorithm: A **number line model** conveniently shows any positive + any positive, any positive + any negative, any negative + any positive, and any negative + any negative. The algorithm becomes "starting at the position representing the first number, use the second number to indicate a distance and a direction for the second move." (Students may notice patterns such as "positive + positive = positive, " negative + negative = negative," and "positive + negative" can only be determined by deciding which of the moves will make the greater impact. See "Absolute Value" below.)

Number Line Model for addition:

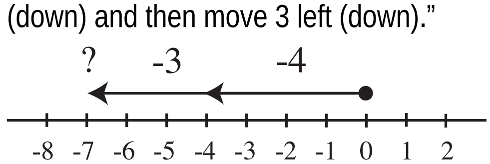
Compute: 4 + -3 = ?

This can be modeled as "start at 0, move 4 right (up), and then move 3 left (down)" and the result is 1.



Evaluate: -4 + -3 = ?

This can be modeled as "start at 0, move 4 left (down) and then move 3 left (down)."



Subtraction Algorithm: A **number line model** works well if the context is "find the difference." (See above for an example.) A **chip board** model works well for "take away" contexts. In the chip board model the algorithm becomes "create a display for the first number that permits the "take away" action of the second number. (Students may try to write patterns such as "positive – positive = ?," or "negative – negative = ?" But they should realize that there are no hard and fast rules here, since the result depends on the absolute values of the numbers involved.

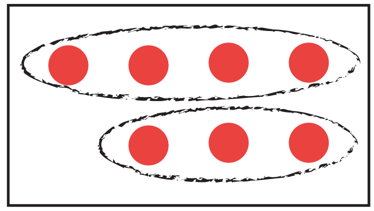
Absolute Value: The distance between a number and zero, regardless of direction. This is useful in thinking about the result of an addition such as +4 + (-7). If we think of these numbers as distances moved, we have moved 4 in a positive direction and 7 in a negative. Since the distance moved in a negative direction is further than the distance moved in a positive direction the result is -3. Absolute value of -7 is 7. Absolute value of +4 is 4. The addition of these two integers is the difference between the absolute values. The notation is |-7| = 7. |-7| is greater than |4| by 3.

+4 + -7 = -3.

Chip Board model for addition:

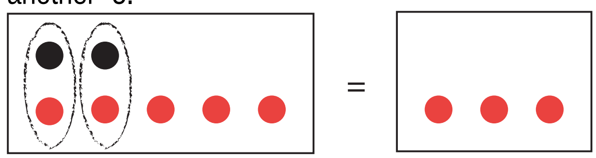
Compute: -4 + -3 = ?

Starting with -4 represented on the chip board, add another -3. Result is -7.



Evaluate: 2 + -5 =?

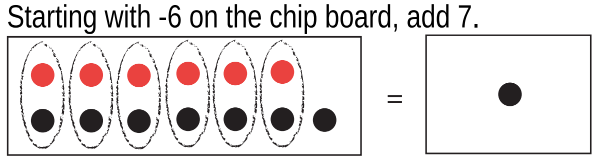
Starting with 2 represented on the chip board, add another -5.



Notice that 2 and -2 are opposites. They are additive inverses. The result of adding 2 + (-2) is zero. So this part of the display can be removed. The result is -3.

Find a value for: -6 + 7 = ?

Starting with -6 on the chip board, add 7.

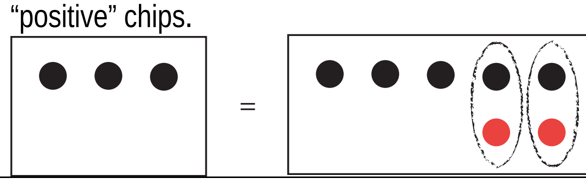


Notice that -6 and +6 are additive inverses. So -6 + (+6) = 0. We can remove this part of the display. So the result is 1.

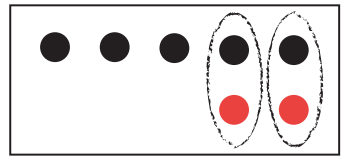
Chip Board model for subtraction:

3 – 5 = ? (*Positive 3 take away positive 5.)*

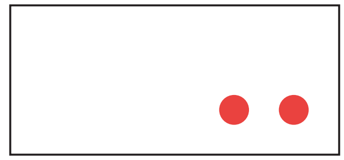
Starting with +3 on the board, we see that we can not subtract +5. In order to accommodate this "take away" we need to alter the initial display so that the starting value is still +3, but there are 5 "positive" chips.



Confirm that the above board does actually represent +3, and then "take away" (+5). The result is -2.

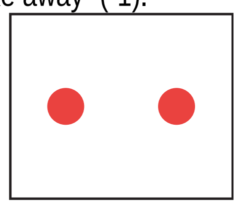


Take away (+5)
Result is



What we have done is equivalent to rewriting 3 – 5 as 3 + (2 – 2) – 5 = (3+2) – 2 – 5 = 5 -2 – 5 = -2.

Compute: -3 – (-1) = ? (Negative 3 take away negative 1.)
Starting with -3 on the chip board we see that we can in fact "take away" (-1).



The result is -2.

3 – (-1) = ? (Positive 3 take away negative 1.)
Starting with (+3) on the chip board we see that we do not have any "negatives" in sight to "take away." Therefore, we alter the display so that the starting value is still 3, but there is one negative chip on the board. In order to do this we have to add 1 positive and 1 negative. Since these are **additive inverses** we actually have added 0.

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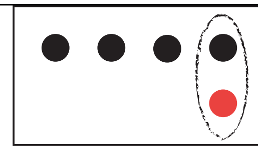
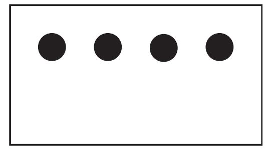


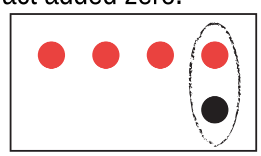
Take away (-1)
Result is



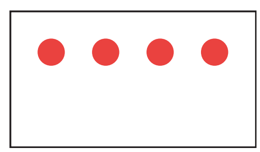
Confirm that the starting value of the above display is actually +3. We can now take away -1. The result is +4.

What we have done is equivalent to rewriting
3 – (-1) as 3 + (1 + -1) – (-1) = 4 + (-1) – (-1) = 4.

Evaluate: -3 – 1 =? (Negative 3 take away positive 1.)
Starting with -3 on the board we see that we do not have any positive chips, so we can not remove 1 positive chip. So we alter the display so that the starting value is still -3 but we can see a negative chip. If we add 1 positive and 1 negative chips we have in fact added zero.



Take away (+1)
Result is



Taking away +1 we have the result -4.
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Algorithm for rewriting subtraction as addition: There is a pattern that connects addition and subtraction and permits every subtraction to be rewritten as an addition. This pattern becomes apparent on the chip board.

A – B = A + (-B), or A minus B is the same as A
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Rewrite the above subtractions as additions:
As above, 3 – 5 = -2
We get the same result if we ADD 3 + (-5) = -2.

As above, 3 – (-1) = 4.
We get the same result is we ADD 3 +(+1) = 4.
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plus the opposite of B.	*As above, -3 – 1 = -4.* We get the same result if we ADD -3 + -1 = -4. *Rewrite -10.5 – (-3.1) as an addition.* -10.5 – (-3.1) is the same as -10.5 + (+3.1) = -7.4.
Multiplication Algorithm: Thinking of multiplication as **repeated addition** we can find answers for a *positive times a negative*. Repeated addition is not so helpful when thinking of a negative times a negative. By using repeated addition, **and patterns** students find that • the product of two positive numbers is always positive • the product of a negative and a positive (in any order) is always negative • the product of two negatives is always positive. Division Algorithm: Using the idea of a **fact family** we have that every division can be thought of in terms of a related multiplication. Thus, -24 ÷ 8 = ? is related to (8)(?)= -24. Thus we see that the missing factor is -3, so -24 ÷ 8 = -3. -36 ÷ -9 = ? is related to (-9)(?) = -36. Thus we see that the missing factor is +4, so -36 ÷ -9 = 4. Using this pattern students find that • the quotient of two positives is a positive • the quotient of a negative divided by a positive is a negative • the quotient of a positive divided by a negative is a negative • the quotient of a negative by a negative is a positive.	*Evaluate 5(-4)* 5(-4) can be thought of as finding the result of placing -4 on a chip board 5 times. The result is -20. *Evaluate -4(5)* We also know that multiplication is **commutative**, and so -4(5) is the same as 5(-4). So -4(5) is also -20. *Evaluate -4(-5).* Using the same logic as above we have the **pattern**, -4(5) = -20 -4(4) = -16 -4(3) = -12 -4(2) = -8 Notice that the result of this multiplication is **increasing** by 4 each time, so continuing the pattern would require us to write -4(1) = -4 -4(0) = 0 -4(-1) = 4 -4(-2) = 8 -4(-3)= 12 -4(-4) = 16 -4(-5) = 20. Notice that the result of -4(5) is the opposite of -4(-5).

Properties of Numbers:

Certain patterns always work for some operations with integers (and rational numbers).

- We can multiply (or add) rational numbers in any order, and thus we say that multiplication (and addition) have the Commutative Property.
  A + B = B + A.   (A)(B) = (B)(A)
- If only multiplication (or addition) are present in a computation with rational numbers then we can group the numbers in any way, and thus we say that multiplication (or addition) are Associative.
  A + (B + C) = (A + B) + C and
  (A)[(B)(C)] = [(A)(B)](C)
- The Distributive Property is only true for multiplication by one rational number over an addition (or subtraction) of several rational numbers.
  A(B + C) = (A)(B) + (A)(C)

Order of Operations:

In order to have consistency of results there are certain conventions that are followed when evaluating computations. These conventions allow everyone to know what is intended by a mathematical expression. In general, one performs any *operations* that are in parentheses first. After that the expression is inspected to see if there are any exponents. Since exponents are like multiplications that have been grouped together we do them next. After exponents we attend to multiplications OR divisions, in order from the left. Lastly we attend to additions and subtractions, again in order from the left.

- Parentheses
- Exponents
- Multiplication or Division from left

Addition is a Commutative Operation:
3 + (-2) gives the *same* result as -2 + 3.
-5 + (-3) gives the *same* result as -3 + (-5).
Notice that subtraction is NOT a Commutative operation. 3 – (-11) = 3 + (+11) = 14. BUT
-11 – 3 = -11 +(-3) = -14. *Reversing the order of a subtraction gives the opposite result.*

Multiplication is a Commutative Operation:
5(6) gives the *same* result as 6(5).
3(-7) gives the *same* result as -7(3).
Notice that division is NOT a Commutative operation. -12 ÷(-4) = 3. BUT -4÷ (-12) = $\frac{1}{3}$.
Reversing the order of a division gives reciprocal results.

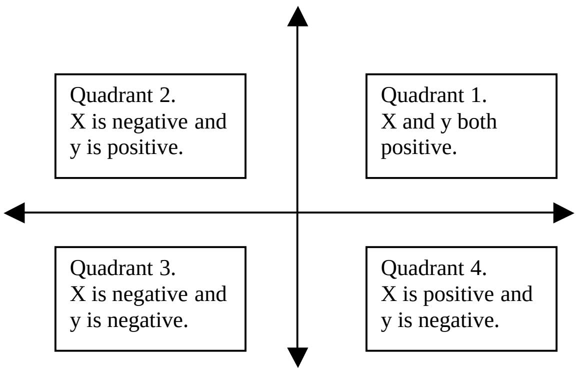
Addition is an Associative Operation:
[4 + (-3)] + (-9) = [1] + (-9) = -8. This is the *same* result as 4 + [(-3) + (-9)] = 4 + [-12] = -8.
Notice that the square brackets in the above example indicate which operation is intended to be done first. (See Order of Operations)
Subtraction is NOT Associative.
[4 – (-3)] – (-9) = [7] – (-9) = 7 +9 = 16. BUT.
4 – [(-3) – (-9)] = 4 – [(-3) +9] = 4 – [6] = -2.

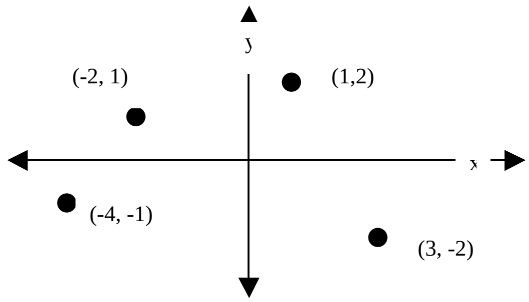
Multiplication is an Associative Operation:
3(-2)(-5)may be evaluated by grouping as [3(-2)](-5) to give (-6)(-5) = 30, OR, by grouping as 3[(-2)(-5)] = 3[10] = 30.

Division is NOT Associative. With no parentheses to guide us 12 ÷ 3 ÷ 4 must be evaluated by doing the divisions from the left. (See Order of Operations.) Thus, 12 ÷ 3 ÷ 4 = 4 ÷ 4 = 1. If we group by inserting parentheses we have
(12 ÷ 3) ÷ 4 = 4 ÷ 4 = 1. BUT if we group differently then 12 ÷ (3 ÷ 4) = 12 ÷0.75 = 16.

• Addition or subtraction from left	Multiplication is Distributive over Addition or Subtraction: 4(3 + 7) = 4(10) = 40. We get the *same* result if we distribute the multiplication by 4 over the two numbers in the parentheses. Thus, 4(3) + 4(7) = 12 + 28 = 40. -4(2 – (-5)) = -4(7) = -28. This is the *same* result as -4(2) – (-4)(-5) = -8 – (20) = -28. Sometimes one format is easier to evaluate than another. For example, $25(\frac{4}{25} + \frac{1}{5})$ is easily evaluated as $25(\frac{4}{25}) + 25(\frac{1}{5})$ = 4 + 5 = 9. Notice that multiplication is not distributive over division, for example. 4(12 ÷2) = 4(6) = 24. This is NOT the same as 4(12) ÷4(2) = 48 ÷ 8 = 6. Applying the Order of Operations: *Evaluate:* $83 - 3(4^3 – 100) + 12 ÷3 \times (-2)^2$ $83 - 3(4^3 – 100) + 12 ÷3 \times (-2)^2$ $= 83 – 3(64 – 100) + 12 ÷3 \times (-2)^2$ (doing one of the operations inside a parentheses first.) = 83 – 3(-36) + 12 ÷3 x 4 (doing another parenthetical operation and an exponent next) = 83 – (-108) + 4 x 4 (doing multiplications or divisions from the left) = 83 – (-108) + 16 (doing the remaining multiplication) = 83 + 108 + 16 = 207.
4 Quadrants of Coordinate Plane: By thinking of the horizontal and vertical axes as two number lines, and extending these number lines so that negative and positive numbers appear, we divide the plane into 4 quarters (quadrants). In one of these quadrants both x and y values are positive. In one of these quadrants the x values are positive and the y values are negative. In one of these quadrants the y values are positive and the x values are negative. In one of these quadrants both x and y values are	*Place the points (1, 2), (3, -2), (-2, 1), (-4, -1) in the correct quadrants.*

negative.


Quadrant 2.
X is negative and
y is positive.
Quadrant 1.
X and y both
positive.
Quadrant 3.
X is negative and
y is negative.
Quadrant 4.
X is positive and
y is negative.


(-2, 1)
(1,2)
(-4, -1)
(3, -2)