Vocabulary: *Bits and Pieces II*

Concept

Example

Estimation: Students have developed **benchmark** fractions. They can use these to substitute for all the fractions in a computation. Or they may use a combination of benchmarks and **equivalence** to estimate the result of computations. In some cases purposeful overestimates or underestimates will cause students to adjust the fractions they choose to work with. By developing a sense of fractions and operations students acquire a way to know if their computations are wrong, whether the calculation has been done by hand or by calculator.

Estimate $\frac{8}{17} + \frac{5}{6}$:

$\frac{8}{17}$ is close to but less than a half. $\frac{5}{6}$ is closer to 1 than to a half. Using these as **benchmarks**, a good estimate would be 1 and a half, but this will be an overestimate.

$\frac{8}{17} + \frac{7}{8} + 1\frac{2}{9}$.

Students might first substitute a half for $\frac{8}{17}$ and $1\frac{1}{4}$ for $1\frac{2}{9}$, to get $\frac{1}{2} + \frac{7}{8} + 1\frac{1}{4}$. They may then use **equivalence** to rename the resulting fractions:

$\frac{4}{8} + \frac{7}{8} + 1\frac{2}{8} = \frac{4}{8} + \frac{7}{8} + \frac{10}{8} = \frac{21}{8} = 2\frac{5}{8}$.

For the same example, if an **overestimate** is practical: $\frac{1}{2}$ + 1 + 1 and a half = 3.

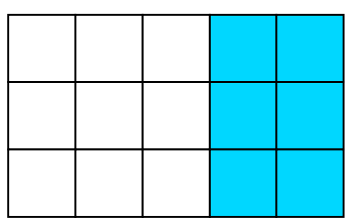
Addition and subtraction: Students may use various models as they make sense of putting together and taking apart fractions. The goal is to develop a mathematically logical algorithm that students understand, which may or may not be the standard algorithm.

- ✓ They may use a **renaming** strategy (**equivalent** fractions) with an **area model**. This involves the use of a **common denominator**.
- ✓ Or they may use a renaming strategy with **fraction strips**.
- ✓ If they are working with **mixed numbers** they may use a strategy that uses renaming and a number line.
- ✓ Or they may work completely with the symbols, renaming as needed.

Using an area model or fraction strip or number line makes clear the logic behind choosing to rename with common denominators.

Starting with the computation $\frac{1}{3} + \frac{2}{5}$ a student might first partition a **rectangular area** into thirds.

Then they might partition the rectangle into fifths, using the other dimension.

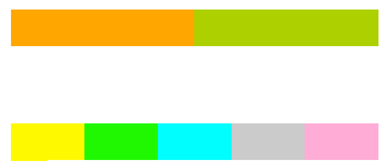


Now the goal is to add the two pieces, representing $\frac{2}{5}$ and $\frac{1}{3}$. The partitioning of the area into thirds and then fifths creates pieces that are $\frac{1}{15}$ths of the area, and each of the target pieces can be **renamed** in terms of $\frac{1}{15}$ths.

$\frac{5}{15} + \frac{6}{15} = \frac{11}{15}$.

The same idea, renaming as fractions with **common denominators**, arises from the use of **fraction strips** which can be folded into various size pieces.

$\frac{1}{2} + \frac{2}{5}$:



Comparing and refolding for $\frac{1}{10}$ths makes the addition possible.

$\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$.

The goal is to make sense of the strategy of **renaming with common denominators**, so that this becomes an efficient and sensible algorithm, which can be used without the supporting models.

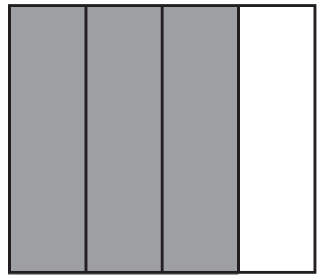
Multiplication: Students must come to the realization that the fraction you multiply by acts as an operator**. If the operator is larger than 1 then the result is an increase; if the operator is smaller than 1 then the result is a decrease**. They also have to make sense of the use of multiplication to model "of," as in "a third of 2 and a half." As before, the goal is to develop efficient algorithms that make sense to students.

- They may use an **Area model**, where length and width are fractions and the area is the product of length and width.
- Or they may **partition a number line**.
- Or, when appropriate, they may use discrete objects to model
- In the case of mixed numbers some

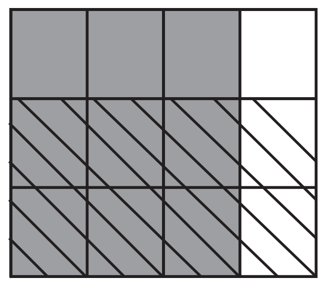
Using an **Area Model**:

Consider the problem $\frac{2}{3} \times \frac{3}{4}$. First represent the $\frac{3}{4}$ by dividing a square vertically into fourths and shading three of the fourths.

students may use the **Distributive Property** to compute a product.



To represent taking $\frac{2}{3}$ of $\frac{3}{4}$, first divide the whole into thirds by cutting the square horizontally, then shade two of the three sections. The part where the shaded sections overlap represents the product, $\frac{6}{12}$.



Note:
This could have been done by noticing that there are 3 shaded areas in the first diagram, so $\frac{2}{3}$ of this would be 2 shaded areas, or $\frac{2}{4}$. This shortcut is not always possible. The first method above, **repartitioning so that the new pieces have a denominator that is a multiple of the denominators of the fractions being multiplied always works.**

Using a **number line:**
If the problem is to evaluate $\frac{1}{3}$ of 2 $\frac{1}{2}$, drawing 2 $\frac{1}{2}$ first, and partitioning into halves only produces 5 pieces. We need a multiple of 3 so that we can take a third.

Marking off the strip or number line into $\frac{1}{6}$ths makes 15 pieces, so we can take one third of these 15 pieces. $\frac{1}{3}$ of 2 $\frac{1}{2}$ = $\frac{5}{6}$. Again, a

common multiple of the denominators determines the size of the pieces in the renaming of the second fraction, and makes the division of the numerator possible.
Note: in both examples so far the multiplier was less than one so the result was less than the second fraction.

Students can **generalize** the above strategy and apply it without drawing pictures:
$\frac{1}{3}$ of $2\frac{1}{2}$ = $\frac{1}{3}$ of $\frac{5}{2}$ = $\frac{1}{3}$ of $\frac{15}{6}$ = $\frac{5}{6}$.
This strategy works by **renaming the second fraction** so that the numerator can be divided by 3, the denominator of the first fraction.

This strategy can be extended to any fraction multiplication:
$1\frac{1}{5}$ of $3\frac{1}{3}$?
Thinking first of $\frac{1}{5}$ of $3\frac{1}{3}$ we get $\frac{1}{5}$ of $\frac{10}{3}$ = $\frac{2}{3}$ (no renaming necessary this time)
So $\frac{6}{5}$ of $3\frac{1}{3}$= 6($\frac{2}{3}$) = $\frac{12}{3}$ = 4.
Note: The multiplier this time was greater than 1 so the result was greater than the second fraction.

The strategy works for **multiplying mixed numbers**: $1\frac{2}{3}$ of $3\frac{1}{5}$= $\frac{5}{3}$ of $\frac{16}{5}$= $\frac{5}{3}$ of $\frac{48}{15}$ = 5($\frac{1}{3}$ of $\frac{48}{15}$) = 5($\frac{16}{15}$) = $\frac{80}{15}$ = $5\frac{1}{3}$.
Students may notice that they could shortcut this by moving straight from $\frac{5}{3}$ of $\frac{16}{5}$ to $\frac{80}{15}$, **multiplying the numerators and denominators.**

The last computation could also be done by using the **Distributive Property**:
$1\frac{2}{3}$ of $3\frac{1}{5}$= $1\frac{2}{3}$ of (3 + $\frac{1}{5}$) = $1\frac{2}{3}$ of 3 + $1\frac{2}{3}$ of $\frac{1}{5}$ = 5 + $\frac{5}{3}$ of $\frac{3}{15}$ = 5 + $\frac{5}{15}$ = $5\frac{1}{3}$.

Division: In part the reason that division needs several approaches is that it has different meanings in different contexts. One might have a situation where a given total has to be divided or **shared or partitioned** into a known number of

If we have 2 yards of ribbon and we want to cut it into lengths of $\frac{1}{6}$ of a yard, how many pieces will we have? (This is a **grouping** question.)

subgroups, answering the question, "how much will each get?". Or the situation might involve a given total to be **divided or grouped** into subsets of a given size, answering the question, "how many groups can be made?" From these situations different, mathematically logical algorithms arise:

ii) Multiply by the denominator and divide by the numerator.
iii) Multiply by the reciprocal.
iv) The common denominator approach.

$2 \div \frac{1}{6} = 12$, because there are $\frac{6}{6}$ in 1 whole yard.

If we have 2 yards of ribbon and we want to cut it into lengths of $\frac{5}{6}$ of a yard we can think first of $2 \div \frac{1}{6} = 12$ and then realize that we will get $\frac{1}{5}$ as many pieces since each is 5 times longer. So we get $\frac{12}{5}$ pieces (or $2\frac{2}{5}$ pieces of ribbon).

The above strategy pays attention to denominator of the second fraction first, and then the numerator. **It appears that we multiply by the denominator of the dividing fraction and then divide by the numerator.**

Another strategy uses **common denominators**:
$\frac{4}{3} \div \frac{1}{15} = \frac{20}{15} \div \frac{1}{15} = 20$.
Extending this;
$\frac{4}{3} \div \frac{3}{15} = \frac{20}{15} \div \frac{3}{15} = (\frac{20}{15} \div \frac{1}{15}) \div 3 = \frac{20}{3}$. Notice this is the same as $\frac{60}{9}$, so the result is the same as **multiplying by the denominator of the second fraction and dividing by the numerator.**

This strategy works for **mixed numbers**:
$1\frac{1}{3} \div 1\frac{3}{5} = \frac{4}{3} \div \frac{8}{5} = (\frac{4}{3} \div \frac{1}{5}) \div 8 = (\frac{20}{15} \div \frac{3}{15}) \div 8 = (\frac{20}{3}) \div 8 = \frac{20}{24}$.
Note: The "common denominator" strategy is not logically different from the "multiply by the denominator and divide by the numerator" strategy.

Inverse relationships, fact families:
Students are familiar with fact families involving addition and subtraction, and multiplication and division, from their elementary school experience. This extends to their work with fractions,

Students already know that 2 + 3 = 5 is related to 5 – 2 = 3 and 5 – 3 = 2. They extend this to:
$\frac{2}{3} + \frac{4}{5} = \frac{22}{15}$ or $1\frac{7}{15}$, so $1\frac{7}{15} - \frac{4}{5} = \frac{2}{3}$ and $1\frac{7}{15} - \frac{2}{3} = \frac{4}{5}$.

Students already know that 2 x 5 = 10 so 10 ÷ 2 = 5 and 10 ÷ 5 = 2. They extend this to:
$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$, so $\frac{8}{15} \div \frac{2}{3} = \frac{4}{5}$ and $\frac{8}{15} \div \frac{4}{5} = \frac{2}{3}$. This foreshadows work done in solving equations like:
$\frac{2}{3}(N) = \frac{8}{15}$.