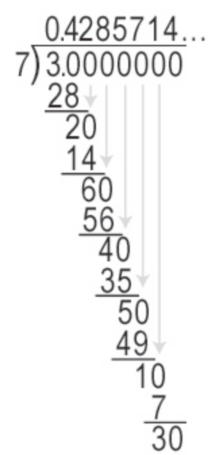
Vocabulary: *Bits and Pieces III*

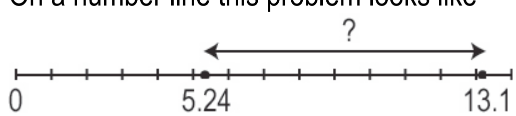
Concept	Example

Meaning of decimal There are two ways to think of a decimal: As a number whose value is signaled by **place value**, or as a representation of a **fraction.**	1. 43 means 4 tens and 3 units. 4.3 means 4 units and 3 tenths. 5.43 means 5 units, 4 tenths and 3 hundredths, or 5 units and 43 hundredths. 1. *Which makes sense: 0.43 + 1 = 0.431, or 0.43 + 1 = 1.43, or 0.43 + 1 = 0.44?* Students who understand place value will know that 0.43 + 1 must be greater than 1 unit. Thus, 1.43 is the only sensible answer. Students who make errors implied in the other two incorrect answers are probably "lining up" the decimal numbers incorrectly because they are not attending to place value. "1" is 1 unit and is placed in the position immediately before the decimal point, as below. Correct Setup: $\begin{array}{rl} & 0.43 \\ + & 1 \\ \hline & 1.43 \end{array}$ Incorrect Setups: $\begin{array}{rr} & 0.43 \\ + & 1 \\ \hline & 0.44 \end{array} \qquad \begin{array}{rr} & 0.43 \\ + & 1 \\ \hline & 0.431 \end{array}$
Decimal forms Some decimals are **terminating** and some are **repeating,** and some neither repeat nor terminate. Students can learn to predict which fractions (**rational numbers**) have decimal representations that repeat and which have decimal representations that terminate. (A later unit, *Looking For Pythagoras*, introduces **irrational** numbers.)	1. A positive **rational number** is any number that can be written in the form $\frac{a}{b}$ where *a* and *b* are whole numbers, and b is not zero. For example: $\frac{2}{3}$, $\frac{5}{2}$, $\frac{4}{1}$, $\frac{0}{7}$, $\frac{3}{8}$, $\frac{2}{13}$ are all rational numbers. 1. Some rational numbers are represented by **terminating** decimals, for example: $\frac{1}{2}$ = 0.5, $\frac{3}{4}$ = 0.75, $\frac{1}{8}$ = 0.125, $\frac{3}{25}$ = 0.12. In each case there may be several decimal places necessary to find the equivalent of a

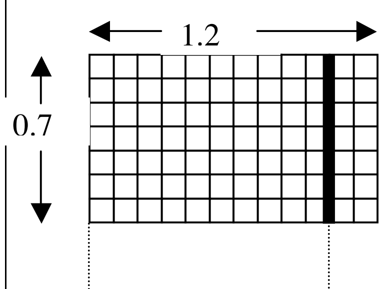
	given fraction, but the number of decimal places needed to make an exact equivalent is finite. 1. Some rational numbers are represented by infinite **repeating** decimal forms, for example: $\frac{2}{3}$ = 0.666666666…, $\frac{8}{15}$ = 0.533333….., $\frac{3}{7}$ = 0.42857142857142…. In each case a pattern emerges after a set number of decimal places and that pattern of digits repeats infinitely. (Note: often people estimate $\frac{1}{3}$ as 0.33. These are **not** exact equivalents.) 1. An example of a decimal that neither repeats nor terminates would be 0.121221222122221….. The decimal representation of $\sqrt{2}$ starts 1.414… but never terminates nor repeats. 1. *How can we predict whether a particular fraction will have a decimal equivalent that repeats or terminates*? Since we know that every terminating decimal can actually be represented by a fraction with a denominator of 10 or 100 or 1000 or some other power of 10 (using the place value notion), we can deduce that every fraction that can be rewritten as a fraction with a denominator that is a **power of 10** will be represented by a terminating decimal. Since the only factors of 10 are 2, 5 and 10, this means that if the denominator of a fraction has only factors of 2 and 5 and 10 then we will be able to rewrite the fraction with a power of 10 as a denominator. For example, $\frac{3}{20}$ can be rewritten as $\frac{15}{100}$ or 0.15; $\frac{7}{125}$ can be rewritten as $\frac{56}{1000}$ or 0.056; $\frac{36}{225}$ can first be rewritten as $\frac{4}{25}$

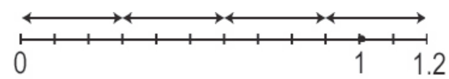
	and then as $\frac{16}{100}$ or 0.16. However, the decimal for $\frac{7}{30}$ will not terminate because 30 has factors other than 2, 5 and 10. 1. *Why does it make sense that every fraction will be represented by either a decimal that terminates or repeats, and not one that neither terminates nor repeats*? In Bits and Pieces II students learned that one of the ways to think of a fraction is as a division. Thus $\frac{3}{7}$ can be thought of as 3 divided by 7. As you can see below, the division process starts with $30 \div 7$, which gives 4 remainder 2, and after 5 further iterations of the process we are back to $30 \div 7$, so the whole process repeats. Since dividing by 7 necessarily can only involve remainders of 0, 1, 2, 3, 4, 5, or 6, we are bound to have repeated remainder eventually, and as soon as the remainder repeats the whole process repeats. 
Estimation As with fraction computations there are productive strategies for making an estimate. These are **benchmark fractions and their decimal equivalents,** and **rounding.** Fraction benchmark ideas can be used to estimate small decimal computations. Students have	1. Using **fractional benchmarks**: 0.78 + 0.14 is near $\frac{3}{4}$ + $\frac{1}{8}$ or $\frac{6}{8}$ + $\frac{1}{8}$. So a reasonable estimate is $\frac{7}{8}$, which is a little less than 1 or perhaps about 0.9.

developed the understanding that 0.5 is equivalent to $\frac{1}{2}$, that 0.75 is equivalent to $\frac{3}{4}$, that 0.33… is equivalent to $\frac{1}{3}$, that 0.666…is equivalent to $\frac{2}{3}$, and that 0.125 is equivalent to $\frac{1}{8}$. They can use these equivalents to simplify the computation, and to predict a reasonable answer. Rounding can also be used to make simpler decimal computations.	$0.78 - 0.14$ would be about $\frac{6}{8} - \frac{1}{8}$ or $\frac{5}{8}$. So a reasonable estimate is about 0.6. 0.78×0.14 would be about $\frac{6}{8} \times \frac{1}{8}$ or $\frac{6}{64}$. So a reasonable estimate is about a tenth. $0.78 \div 0.14$ would be about $\frac{6}{8} \div \frac{1}{8}$ or about 6. 10. Using **rounding** techniques, we may decide to work with only 1 decimal place. If the digit in the second decimal place ($\frac{1}{100\text{'s}}$) is greater than or equal to 5, then we round up, if not we round down. Thus 0.78 would be rounded up to be 0.8, while 0.74 would be rounded down to be 0.7. $0.78 + 0.14$ is near $0.8 + 0.1$ but a bit greater than 0.9. $0.78 - 0.14$ would be near $0.8 - 0.1$ but a bit less than 0.7. 0.78×0.14 would be about 0.8×0.1 or a bit greater than 0.08. So you might guess that the product is close to 0.1. $0.78 \div 0.14$ would be about $0.8 \div 0.1$ or about 8. Since the dividend was rounded up and the divisor was rounded down, the estimate will be too large. So we might estimate the answer to be closer to 6.
Addition Subtraction of Decimal Numbers: Students have to interpret, model and symbolize problems involving decimal computations. One way that they can think about decimal addition or subtraction is to concentrate on **place value**. Another way is to change the representation to **fractions with**	11. *Lisa is walking in a charity marathon. The entire course is 13.1 miles and Lisa's pedometer tells her she has walked 5.24 miles already. How much further has she to walk?* Students have to be able to recognize that the language in this problem signals a subtraction of one part from the

common denominators. Additions and subtractions are often represented on a **number line.**	whole to find the other part. Thus, we have to compute 13.1 – 5.24. • Using **place value** we know that the “5” has to be placed under the “3” so that the units are lined up, and the “2” has to be placed under the “1” so that the tenths are lined up. Once this is done then subtraction algorithms learned in elementary school apply. 13.10 –5.24 7.86 • Using a fraction representation we know that 13.1 means the same as 13 and $\frac{1}{10}$, while 5.24 means the same as 5 and $\frac{24}{100}$. We have to compute $13\frac{1}{10} - 5\frac{24}{100} = 13\frac{10}{100} - 5\frac{24}{100}$. One way of proceeding from there is to rewrite this as $12\frac{110}{100} - 5\frac{24}{100}$ $= 7\frac{86}{100}$ or 7.86. On a number line this problem looks like 
Multiplication of Decimal Numbers: Students use the **fractional representation** of decimals to find answers for multiplication of decimals. From the fractional representation they develop a **shortcut algorithm** for multiplication of decimals. Students should be able to predict an approximate answer so that errors with the algorithm can be avoided.	12. Using a **fraction representation** and skills developed in *Bits and Pieces II*: • $1.3 \times 2.4 = \frac{13}{10} \times \frac{24}{10}$. From here students might have an algorithm that does the $\frac{1}{10}$ of $\frac{24}{10}$ first to get $\frac{24}{100}$ and then multiplies by 13 to get $\frac{13\times24}{100}$. However their algorithm from BPII has been understood they will end with

Impact of multiplying by decimals: In elementary school when students multiplied one whole number by another the answer was always greater than either of the factors being multiplied. This might lead students to internalize a rule that multiplication always leads to an answer greater than either factor being multiplied. Yet this rule only works with positive whole numbers. When students use fraction representations of decimal multiplications they can make sensible estimates and can predict how the answer will compare to either of the factors being multiplied. An area model helps make the comparison clear also.	$1.3 \times 2.4 = \frac{13 \times 24}{100} = \frac{312}{100} = 3.12$ (using an understanding of place value). • $1.3 \times 0.24 = \frac{13}{10} \times \frac{24}{100} = \frac{312}{1000} = 0.312$ • $0.13 \times 0.24 = \frac{13}{100} \times \frac{24}{100} = \frac{312}{10000} = 0.0312$ 13. *By studying the pattern of answers in example 12, can we predict the digits in the final answer and the place values of those digits?* We can see that the digits involved are always "312." This is because the fraction operation always involves multiplying "13" and "24." The place values depend on the place values of the original factors. When we have "$\frac{1}{10\text{'s}}$" to multiply by "$\frac{1}{100\text{'s}}$" we have a result that involves "$\frac{1}{1000\text{'s}}$." When we have "$\frac{1}{100\text{'s}}$" to multiply by" $\frac{1}{100\text{'s}}$" we have a results that involves "$\frac{1}{10000\text{'s}}$." Thus we can predict the number of decimal places needed in the final answer by counting the decimal places in the original problem. 14. *Multiply 1.5×2.4.* Using the shortcut algorithm described in example 13 we know the digits involved will be 15×24 = 360. Since there are "$\frac{1}{10\text{'s}}$" and "$\frac{1}{10\text{'s}}$" in both factors we know the result will involve "$\frac{1}{100\text{'s}}$." Therefore we need 2 decimal places in the answer. 1.5×2.4 = $\frac{360}{100}$ = 3.60. 15. *Multiply 1.5×0.24.* Using the shortcut algorithm we have 1.5×0.24 = $\frac{360}{1000}$ = 0.360. 16. *Will is computing the price of 2.3 pounds of tomatoes at $0.88 a pound. He multiplies 23 by 88 and gets 2024. So he incorrectly charges $20.24. Why does*

	Will's error arise and how might Will think about the problem to avoid this error? Will starts out to use the shortcut algorithm. In this case he has 4 digits, "2024." He knows he has to make dollars and cents out of this, which may have led to his incorrect answer. He has "$\frac{1}{10's}$" and "$\frac{1}{100's}$" in the original problem so he should have "$\frac{1}{1000's}$" in the answer. The final answer should be $\frac{2024}{1000}$ or 2.024 or \$2.02. (If Will had used rounding to estimate this problem he might have used 2×1 instead of 2.3×0.88. Thus, he should be expecting an answer close to \$2.) 17. *Predict whether the result of 1.5×2.4 is greater or less than 2.4.* This is the same as $1\frac{1}{2} \times (2.4)$. Obviously multiplying 2.4 by $1\ \frac{1}{2}$ gives and answer greater than 2.4. 18. *Predict whether the result of 0.15×2.4 is greater or less than 2.4.* This is the same as $\frac{15}{100} \times 2.4$. Obviously multiplying 2.4 by a number less than 1 gives a result less than 2.4.
Models for Multiplication of Decimals: Decimal multiplication is often represented by an **area model**, where the length and width of a rectangle represent the two decimal numbers being multiplied. Another model for multiplication involves **partitioning**.	*19. Use an **area model** to represent 0.7×1.2.*  The length is divided into 2 pieces by a heavy line indicating 1 unit of length. (The broken lines indicate how one would complete 1 square unit.) Each of the small

	squares is $\frac{1}{100}$ of 1 square unit. Therefore, the area representing the answer is made of two pieces, 0.7×1 and 0.7×0.2. Thus, $0.7 \times 1.2 = 0.7 \times 1 + 0.7 \times 0.2 = 0.7 + \frac{14}{100} = 0.84$ of a square unit. 20. *Use a partitioning explanation to find the answer for 0.25×1.2.* $0.25 \times 1.2 = \frac{1}{4} \times 1.2 = 0.3$ as shown below.  $.25 \times 1.2 = 0.3$
Division of Decimals: Division is about dividing a total among equal size groups. Either we know the number of groups and want to know the size of each group, or we know the size of each part and want to know how many parts can be made. Using a **fraction representation with common denominators** students can divide one decimal by another, and can reason about a shortcut algorithm. Students should be able to interpret their answers.	21. *Represent $3.2 \div 0.8$ using fractions with a common denominator and compute the answer.* $3.2 \div 0.8 = \frac{32}{10} \div \frac{8}{10} = 32 \div 8 = 4$. 22. *Represent $3.2 \div 0.08$ using fractions with a common denominator and compute the answer.* $3.2 \div 0.08 = \frac{32}{10} \div \frac{8}{100} = \frac{320}{100} \div \frac{8}{100} = 320 \div 8 = 40$. 23. *Represent $0.32 \div 0.8$ using fractions with a common denominator and compute the answer.* $0.32 \div 0.8 = \frac{32}{100} \div \frac{8}{10} = \frac{32}{100} \div \frac{80}{100} = 32 \div 80 = 4 \div 10 = 0.4$. 24. *In every example in 21 through 23 we started with a decimal division statement and ended with a whole number division statement. In example 21 we ended with $32 \div 8$, while in example 22 we ended with $320 \div 8$. What algorithm tells how to change from a decimal statement to a whole number statement?* In example 21 the common denominator was 10, so we

multiplied both decimals by 10 to make them into a whole number statement. In example 22 the common denominator was 100 so we multiplied both decimals by 100 to make them into a whole number statement.

25. *Use a shortcut algorithm to divide 0.4 by 1.05.* Multiplying both decimals by 100 this can be rewritten as $40 \div 105$. Shown below is a set up for this division that preserves the place value. Notice that the decimal point is placed to keep the "40," and enough zeroes are added to keep the division process going until the decimal answer either repeats or terminates.

$$
\begin{array}{r}
.3\overline{809523}\ldots \\
105\overline{)40.00000} \\
\underline{315} \\
850 \\
\underline{840} \\
1000 \\
\underline{945} \\
550 \\
\underline{525} \\
250 \\
\underline{210} \\
40
\end{array}
$$

26. *The cost per student for a field trip is \$25.40. The class fundraiser produces \$512. How many students can be taken on the field trip?* Since the common denominator is 100 we can rewrite this as $51200 \div 2540$, as shown below.

$$
\begin{array}{r}
20.1 \\
2540\overline{)51200.} \\
\underline{5080} \\
4000 \\
\underline{2540} \\
\text{etc.}
\end{array}
$$

The answer for the division should be interpreted as "20 students" with \$4 left over. (Note: students can make a reasonable prediction by rounding this as $500 \div 25$.)

Percent

Students already know the definition of percent as $\frac{1}{100}$ (See Bits and Pieces II). The percent

27. *Jill wants to buy a CD that is priced at \$7.50. The sales tax is 6%. What will be the total cost of the CD?*

as $\frac{1}{100}$. (See Bits and Pieces II) The percent problems in this unit all require **solving an equation of the format *a*% of *b* equals *c***, in which any one of the letters, *a*, *b*, or *c* may be the missing value.

The equation this time is 6% of \$7.50 = *c.* (We first find the sales tax and then add it on to the price.) Using the definition of percent, we have $0.06 \times 7.50 = c$. As in examples 12 through 18 we can think of this as $\frac{6}{100} \times \frac{750}{100} = \frac{4500}{10000} = 0.4500$ or \$0.45. Or we can proceed straight to the shortcut algorithm. So the total cost is \$7.50 + \$0.45 = \$7.95.

28. *Customers left Jerome \$2.50 as a tip for service. The tip was 20% of the total bill for their food. How much was the bill?*

    The equation this time is 20% of b = \$2.50. Students might reason that if 20% of a sum of money is \$2.50 then 100% must be 5 times as much, or \$12.50. Or they might use fractions and write $\frac{20}{100} \times \frac{b}{1} = \frac{250}{100}$. This means that $20 \times b = 250$. So $b = \frac{250}{20} = 12.5$. (Students only solve algebraic equations with informal methods at this stage. Later they learn to apply properties of equations.)

29. *At another music store, Sam got a \$12 discount off a purchase of \$48. What percent discount did he get?* In this situation our equation looks like this:

    a% of \$48 equals \$12.

    Students can informally solve this by asking themselves how many 12s it takes to get 48. It takes four 12s to make 48 so the percent must be $\frac{1}{4}$ of 100%. This would be 25%. Or they may write

	$\frac{a}{100} \times \frac{48}{1} = \frac{12}{1}$ or $\frac{a}{100} \times \frac{48}{1} = \frac{1200}{100}$. From this they can deduce that $a \times 48 = 1200$, so $a = 25$. 30. *If an advertisement for cat food says that "80 out of 200 cat owners say their cat has bad breath," what percent of cat owners say their cat has bad breath?* The equation is *a%* of 200 = 80. Some students will use fraction thinking to ask what $\frac{80}{200}$ is as a decimal. This gives 0.4 or 40%. Some will write $\frac{a}{100} \times 200 = \frac{80}{1}$ or $\frac{a}{100} \times \frac{200}{1} = \frac{8000}{100}$. So $a = 40$. Note: These informal equation-solving techniques are powerful ways of thinking that are based on understanding the situation. Equation-solving techniques are more fully developed in grades 7 and 8. Rushing to techniques here may mask understanding of the problem situations and what the problem is asking.

Circle Graph The key to a circle graph is that you know there are 360° in a full turn around the center of a circle. To represent the data you need to figure out what the angle is that represents the amount of turn for a certain percent of the data.	31. *If we want to represent 10% of a budget on a circle graph of the total budget, how many degrees would we need to use?* If students understand that 100% of the budget is represented by 360 degrees, then this question is really the equation 10% of 360 = c. $\frac{10}{100} \times 360 = c$ $\frac{1}{10} \times 360 = c$ $36 = c$.