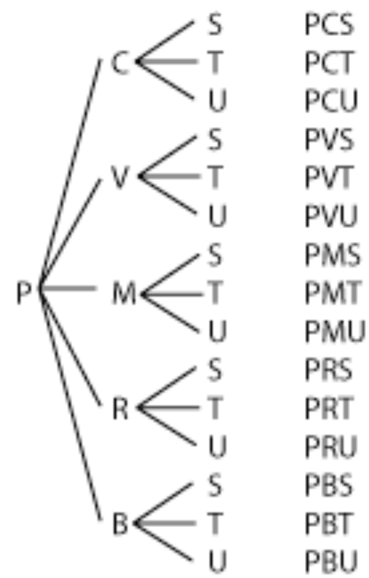
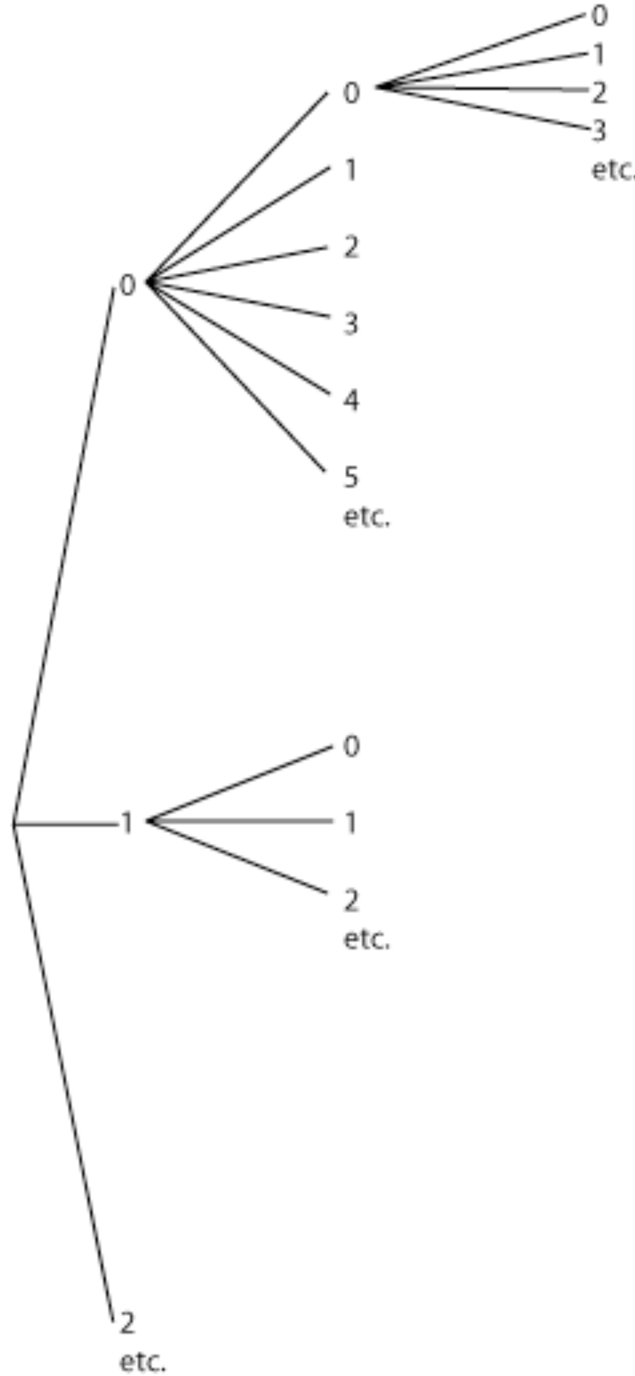
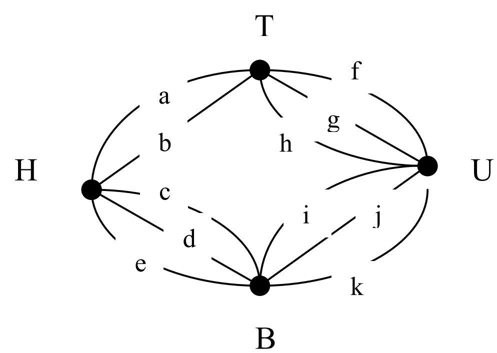
Clever Counting: Sample ACE Solutions
Investigation 1: #3, #6a
Investigation 2: #4, #11
Investigation 3: #10
Investigation 5: #7, 8

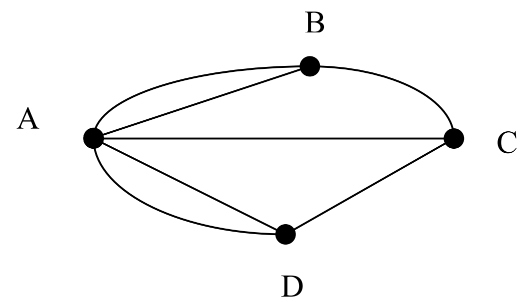
ACE Question	Possible Solution
Investigation 1	
3. At the meeting of Future Teachers of America, ice cream cones were served. There were 2 kinds of cones, five flavors of ice cream, and three types of sprinkles. Each member chose one cone, one ice cream flavor, and one type of sprinkles. How many combinations were possible? Show your work.	3. Let's call the types of cone "plain" and "waffle" or P and W. Let's call the flavors of ice cream "chocolate," "vanilla," "mint," "raspberry," and "bubblegum," or C, V, M, R, B. Let's call the sprinkles S, T and U. Now we can make a list or tree showing the combinations:  We would have another 15 combinations starting with W, so there are 30 ways to assemble an ice cream cone from these choices.
6. License plates in the state where the locker robbery took place contain three letters followed by three numbers. In Problem 1.2, you found that this scheme provides enough plates for over 17 million cars. In states with small populations, such as	6. a. Since there are 26 choices for letters of the alphabet and only 10 choices for numbers, it makes sense to use letters where possible. We can understand how using letters increases the number of choices if we think of a tree diagram. If we use only letters then the first stage in a tree diagram would have 26 choices, and each choice of letter will

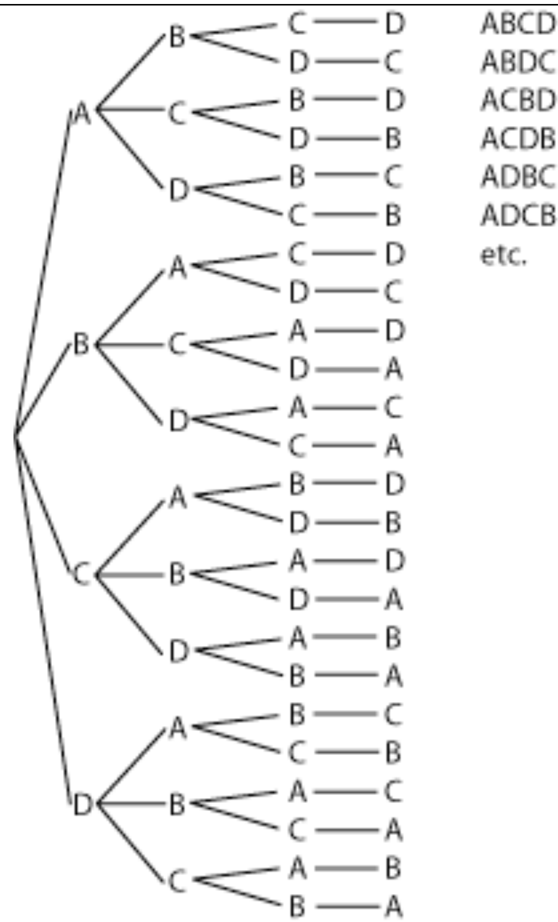
Alaska, North Dakota, Wyoming, and Vermont, fewer than 1 million cars are registered. a. Suppose you are in charge of developing a license-plate scheme for a state with a million registered cars. Describe a scheme that would provide enough plates for all the cars and require the fewest characters. b.	have 26 "branches" coming from it (assuming repeats are allowed). Compare this to using only numbers: there would be 10 choices at the first stage and each choice would have 10 "branches" coming from it. If we use 5 letters we can produce over 11 million combinations for number plates. Using 5 numbers produces only 100,000 combinations. If we use 5 letters and allow repeats we can make over 1 million: for example GOMSU. Or we can make over a million by using 4 letters and 1 number: for example, TREE9. Or by using 3 letters and 2 numbers: for example, FLA22. We need 5 characters in each of these cases. Student solutions might vary if they use rules eliminating some combinations.
Investigation 2	
4. Which of the following has the greatest number of possible combinations? Explain how you arrived at your answer. i. a lock with 10 numbers for which a combination consists of 3 numbers with repeated numbers allowed. ii. a lock with 5 numbers for which a combination consists of 5 numbers with repeated numbers allowed iii. A lock with 5 numbers for which a combination consists of 5 numbers with repeated numbers not allowed.	4. Students might use a tree or a list to analyze this problem. Or they may think of this more abstractly as slots to fill, and numbers of choices for each slot. i. We can analyze this by thinking of a tree, even if the actual tree is unwieldy to draw. The tree would have 3 levels or stages, representing the 3 choices to be made. At the first stage there are 10 choices. At the second stage there are 10 "branches" coming from each of the first choices. This is started below.

	 As you can see this results in (10)(10)(10) = 1000 combinations. ii. We can visualize a tree with 5 stages. At the first stage there are 5 choices to be made. For each of these choices there are 5 “branches” and so on. This results in (5)(5)(5)(5)(5) = 3125 combinations.

	iii. This time the first stage of the tree diagram could have 5 choices, but only 4 "branches" would come from each of these choices for the second stage, because one number has already been used and no repeats are allowed. And at the 3^{rd} stage only 3 choices are left, because 2 numbers have already been used. And so on. This leads to (5)(4)(3)(2)(1) = 120 combinations.
11. a. How many different eight-letter "words" can you make by rearranging the letters in the word COMPUTER? In this situation a "word" is any combination that includes each letter in COMPUTER exactly once. b. Which lock problem does this resemble? Explain.	11. a. If we think of this as having 8 letters to fill 8 slots then we have: __ __ __ __ __ __ __ __ For the first slot we have all 8 letters available to us. But when we turn to the second slot we have only 7 letters left. And when we turn to the third slot there are only 6 letters left. And so on. <u>8 choices</u> <u>7 choices</u> <u>6 choices</u> <u> 5 </u> <u> 4 </u> <u> 3 </u> <u> 2 </u> <u> 1 </u> (You can also think of this as 8 choices at the first stage of a counting tree, and 7 branches from each of those choices etc.) So there are 40,320 combinations. The letters of COMPUTER are all different so each of these 40320 "words" is different. b. Since the letters are all different, and since we can only use each letter once, this is like finding all combinations of 8 letters chosen from a group of 8 letters, with no repeats. This is like the push-button lock problem (2.1) IF the lock has 8 buttons and ALL of them have to be pushed, with no repeats. Or it could be like the combination lock problem IF the lock had 8 marks, and the combination had to use ALL 8 marks and none could be repeated. (2.2) The point of having students compare this problem to lock problems is to have them attend to the essential structure of the problem: how many slots to fill, how many choices for each

	slot, are repeats allowed, does order matter.
Investigation 3	
10. The diagram shows the routes a professor who lives in Detroit, Michigan, could take to the University of Windsor in Ontario, Canada. a. Draw a simple network with nodes and edges to model this situation. b. Use you network to find the number of possible different routes from the professor's home to the university. See student text for the diagram.	10. a. There are 4 nodal points where choices are made. The professor has 5 choices on setting out from home. At the tunnel, heading towards the university, the professor has 3 choices to make. At the bridge the professor has 3 choices to make. On the return journey the professor has 6 choices on leaving the university. In the following diagram, Home, Tunnel, Bridge, and University represent the nodes where choices are made. And the edges represent the roads between these nodes.  A list is helpful here. af, ag, ah, bf, bg, bh, ci, cj, ck, di, dj, dk, ei, ej, ek. There are 15 possible routes from home to university. WE could also get this answer by applying the Fundamental Counting Principle. Through the tunnel there are (2)(3) routes. Through the bridge there are (3)(3) routes. And then we have to add these to get the final answer. c. Not answered here d. Not answered here Note: this problem is reminiscent of the classic problem called the Konigsberg Bridge Problem. This is about a city on a river, with parts of the

	city on 2 islands in the river. There are 7 bridges connecting 4 different parts of the city. The challenge is to determine if a person can visit each of the 4 parts of the city without crossing a bridge more than once. Here's the problem reduced to a network: 
Investigation 5	
7. Akili, Beatrice, Consuelo, and David eat lunch together every day. a. Use the letters A, B, C, and D to represent the students. List all the possible orders in which the four students could stand in the lunch line. b. Make a counting tree for finding all the possible orders in which the four students could stand in line. How many paths are there through the tree? c. If Elena joined the students for lunch, in how many possible orders could the five students stand in line?	7. a. ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA. Note: If we think of this as 4 slots to fill where we have no repeats (and order matters) we can apply the Fundamental Counting principle and get (4)(3)(2)(1)= 24 ways to order A, B, C, D. Note: The technical name for the 24 entries in the above list is *permutations* of the 4 items A, B, C, D, i.e. different orders for the 4 items. When students study this subject of combinatorics at a more advanced level they may use the word *permutations* for combinations of k items chosen from a group of n items, where different orders of the k items are considered different, and reserve the word *combinations* for situations where order does not matter. For example, there is only one way to choose 4 items from the group A, B, C, D , if order does not matter. But there are 24 ways to order A, B, C, D, if each order is considered unique. b. 24 paths as follows:

	 c. Adding Elena would add another choice at the first stage in the tree, and at every succeeding stage. There would be 5 ways to choose the first person in the line, 4 ways to choose the 2^{nd} person, 3 ways to choose the 3^{rd}, 2 ways to choose the 4^{th}, and only one way to choose the last. (5)(4)(3)(2)(1) = 120 ways to order A, B, C, D, E.
8. When counting two-choice combinations, it often helps to make a chart like the one below. Detective Curious made this chart to help her to assign tasks to her detectives. The names of the detectives are listed across the top of the chart, and the tasks are listed	8. a. It helps to think of this as 5 slots to fill. The 5 slots are ➢ Determining lock combinations ➢ Investigating license plates ➢ Conducting interviews ➢ Gathering descriptions ➢ Researching phone numbers

along the side. Each box in the chart represents a possible assignment. The mark indicates that Clouseau has been assigned the task of gathering descriptions.
See student text.

a. In how many ways can Detective Curious make the assignments if each detective is to have a different task?
b. What other method could you use to count the possible ways to assign the tasks?
c. Would this type of chart be useful for counting the umber of different faces a police artist could draw by combining eye, nose, hair, and mouth attributes. Explain.
d. Would this type of chart be useful for counting the number of possible three-number lock combinations?

After the first slot has been filled with a detective's name, that name can not be used again. So, there are 5 ways to fill the 1st slot, 4 ways to fill the 2nd slot, etc.
(5)(4)(3)(2)(1) = 120 ways to fill the 5 slots.
One of these ways is shown below:

	C	H	Jane	S	Jess
1.		X			
2.			X		
3.					X
4.	X				
5.				X	

Making these assignments is exactly the same as writing the 5 detective names in different orders. The order shown in the above table is H, Jane, Jess, C, S. There are 120 different orders.

b. Students might refer to a counting tree or the Fundamental Counting Principle as ways to calculate the different task assignments.
c. We could use a chart like this if we want to find the number of ways to pair up types of eyes and noses, but not if we want to make combinations of more than 2 facial characteristics.
d. We could use a chart like this if the horizontal headings represented the first number and the vertical headings represented the 2nd number. Then each cell in the chart would represent a possible combination of 2 numbers. But if we want three –number combinations then this chart will not work.