Clever Counting: Concept with Explanation

<table>
<tr><th>Concept</th><th>Example</th></tr>
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<td>Organized list a list, usually of combinations, in which the logic in the order makes it possible to check that the list is complete. Some issues to consider in making the list is whether order matters and whether repeats are allowed.</td>
<td>
1.
How many different outfits can we put together from 3 shirts (S, T, U), 3 pairs of pants (P, Q, R), and 2 hats (H, I)?
If we make an organized list we see there are 18 choices. Below are all the outfits that use shirt, S.
<table>
<tr><th>Shirt
S, T, U</th><th>Pants
P, Q, R</th><th>Hat
H, I</th><th>Organized
List</th></tr>
<tr><td>S</td><td>P</td><td>H</td><td>SPH</td></tr>
<tr><td>S</td><td>P</td><td>I</td><td>SPI</td></tr>
<tr><td>S</td><td>Q</td><td>H</td><td>SQH</td></tr>
<tr><td>S</td><td>Q</td><td>I</td><td>SQI</td></tr>
<tr><td>S</td><td>R</td><td>H</td><td>SRH</td></tr>
<tr><td>S</td><td>R</td><td>I</td><td>SRI</td></tr>
</table>
At this point in the list we see that, starting with choice S for a shirt, and choice P for pants, there are 2 ways to complete the outfit, H or I. Likewise starting with choice S for shirt and Q for pants there are 2 ways to complete the outfit. And starting with “SR” there are another 2 ways. This completes all 6 choices, starting with choice S for a shirt.

Now we start with choice T for a shirt and get:
TPH, TPI, TQH, TQI, TRH, TRI. 6 choices.

And lastly we start with choice U for a shirt, and get;
UPH, UPI, UQH, UQI, URH, URI. 6 choices

There 18 choices in all.
2.
How many ways can we choose 4 numbers from 0 to 9, to make a lock combination?
This time we can have repeats, and order does matter. For example, we can have 3, 4, 4, 1 as a combination. And 4, 4, 3,1 would be a different combination. (In fact, when order does matter the technical word for this is a permutation, but the word in everyday use for a lock is combination.)
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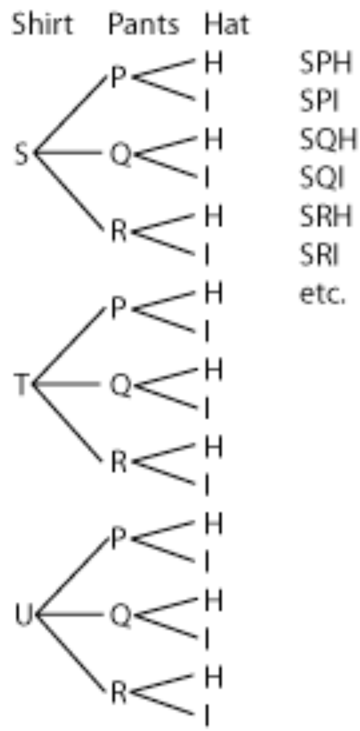
1st choice	2nd choice	3rd choice	4th choice	Organized list
0	0	0	0	0000
0	0	0	1	0001
0	0	0	2	0002
0	0	0	3	0003
0	0	0	4	0003
0	0	0	5	0005
0	0	0	6	0006
0	0	0	7	0007
0	0	0	8	0008
0	0	0	9	0009
0	0	1	0	0010
0	0	1	1	0011
0	0	1	2	0012
0	0	1	3	0013
				etc

The first 10 choices start with 0,0,0, and run through all 10 choices for the 4th position. The next 10 choices start with 0, 0, 1 etc. The list is too long to complete here. But because it is organized we can see how to complete it.
There would be 10 choices starting 002,
10 choices starting 003 etc., so 100 choices starting 00_ _ .
There would be another 100 choices starting 01 _ _. And another 100 choices starting 02 _ _. Etc.
So 1000 choices starting with
0 _ _ _.

There would be another 1000 choices starting 1 _ _ _. And another 1000 starting
2 _ _ _, and so on.

Making 10,000 choices in all. (The choices are, in fact, 0000 to 9999.)

<table>
<tr><td></td><td></td></tr>
<tr><td>Counting Tree a visual way of making an organized list of combinations of options, where the first level indicates the first choice to be made, and the branches indicate succeeding choices, given the first choice has been made.</td><td>3.
We can use a tree diagram as an alternative to the table and list in example 1. Tree diagrams are useful to organize your thinking. For situations where there are many choices the complete tree diagram gets unwieldy.


4.
Suppose a restaurant offers 3 side dishes, from a list of 4: sweet corn, mashed potatoes, coleslaw, hushpuppies. How many ways can a customer choose 3 sides?
Since there is no prohibition on choosing repeats it is possible that a customer would choose coleslaw for all 3 sides. Also we have to consider order. Is coleslaw, sweetcorn and hushpuppies a different choice from hushpuppies, sweetcorn and coleslaw? No. So in this situation, repeats are allowed and order does not matter. The tree diagram below shows there are 16 ways to combine 3 sides, starting with sweet corn, but that some of these are actually the same choices. The redundant combinations have been crossed out, leaving 10 different combinations of side dishes.</td></tr>
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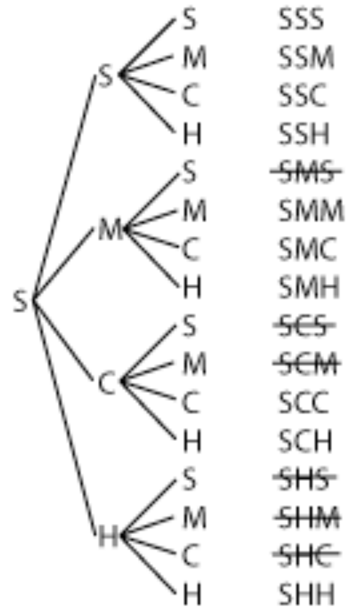
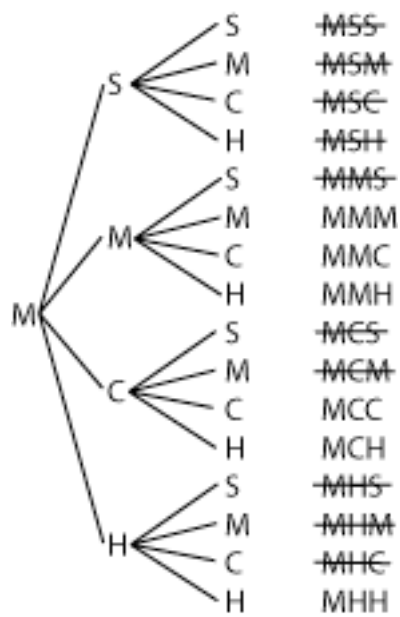


The tree diagram below shows there are 16 ways to combine sides starting with mashed potatoes. But recall that order does not matter, and so any combination with sweet corn has already been accounted for in the first tree diagram, and again some combinations are repeats. This time only 6 different combinations are left.



If you start a new tree diagram with Coleslaw in the first place you should find that only 3 new

	combinations are added. And if you start with hushpuppies only 1 new combination is added. You should find that there is a total of 20 different combinations of side dishes.
Fundamental Counting Principle states that if a task involves a sequence of *k* choices, where n_1 is the number of ways the first stage or event can occur and n_2 is the number of ways the second stage or event can occur after the first stage has occurred, and so one, **then the total number of different ways the task can occur is:** $\mathbf{(n_1)(n_2)(n_3)(n_4)\ldots.(n_k)}$	5. If you examine the organization in example 1 and 3 you can see that there are 3 different slots to fill: <u>Shirt</u>, <u>Pants</u>, <u>Hat</u> Each of these slots can be filled a particular number of ways: <u> 3 </u>, <u> 3 </u>, <u> 2 </u>. Studying the list and tree you can see that the total number of ways to combine these choices is the product of these: (3)(3)(2) = 18 choices. 6. Applying the Fundamental Counting principle to the restaurant side dishes example, #4, we would have to say there are 4 ways to fill each slot since choosing the same dish more than once is allowed: <u>first choice</u>, <u>2nd choice</u>, <u>3rd choice</u>. This would give a total of (4)(4)(4) = 64 choices. But the tree diagram analysis started in example 4 leads to the answer 20 choices. What has gone wrong? Well, the Counting Principle tells us the total number of ways we can combine sides, but does not tell us which of these are actually different. Remember that choosing SMC is the same as choosing SCM. We need a way to eliminate all the combinations that are actually duplicates of other combinations. If we make a list of all 64 choices some will occur 6 times, and some will occur 3 times. So, the Fundamental Counting Principle is only the beginning step in analyzing this problem.
Permutation of k items from a list of n items, where order DOES matter, can be achieved a specific number of ways. For example, choosing 3 students from a group of 10, to perform 3 specific classroom tasks, can be achieved a number of ways. The	7. *How many permutations (combinations) are there of 3 side dishes sweet corn, coleslaw, mashed potatoes if you can not choose the same dish twice and order matters?* This time we are choosing 3 side dishes from 3 side dishes. Making a tree we have:

<table>
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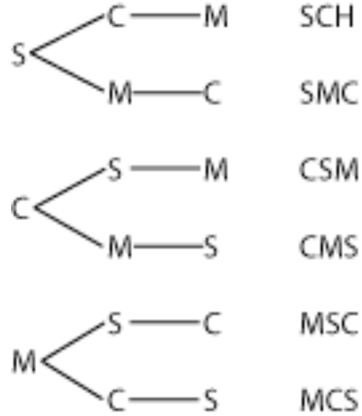
<td>permutations can be listed in an organized list, or on a tree diagram, and then counted.

Note: when order is NOT important the technical word for this is a combination. In the student text for Clever Counting the word Permutation is not used. Instead the text uses Combination for both situations, but states when order is important and when it is not. See below.</td>
<td>

Using the Fundamental Counting Principle we have (3)(2)(1) or 6 ways to combine S, C, and M if we do not use any choice more than once, and if the order matters, that is SCM is considered different from MCS. (In reality we would not consider SCM different from MCS, but the fact that there are 6 ways to permute any three side dishes is helpful in determining how many duplicates there will be among a list of combinations of 3 side dishes if order does NOT matter.)

8.
How many permutations are there of 3 side dishes chosen from a list of 4, if you can not choose the same dish twice and if the order matters?
There will be 4 choices for the first dish, 3 for the 2^{nd}, and 2 for the 3^{rd}.
Using the Fundamental Counting Principle there will be (4)(3)(2) = 24 ways to combine 3 side dishes, if you can not choose the same dish twice, and if the order matters.

Now, putting the ideas from examples 7 and 8 together we can see that there are 24 ways to combine 3 dishes if order matters, but that each selection of 3 dishes will in fact be repeated 6 times, so only 24 ÷ 6 = 4 distinct ways of choosing 3 dishes from 4 actually exist if order does NOT matter.
These are SMC, SMH, SCH, MCH.</td>
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	(Notice that example 4 allowed the same item to be chosen, so the final answer for example 4 was not the same as the answer for example 8.) 9. *How many permutations are there of 3 numbers chosen from 1, 2, 3, 4, 5 if there are no repeats of numbers and order matters?* We have 3 slots to fill. There are 5 choices for the 1^{st} slot, and 4 choices for the 2^{nd} choice, and 3 choices for the 3^{rd} choice. There are (5)(4)(3) = 60 ways to permute 3 numbers chosen from a list of 5. 10. *How many ways can we choose 3 students from a class of 10 to fill 3 class positions (treasurer, secretary, president)?* There are 10 ways to fill the treasurer position, leaving 9 ways to fill the secretary position, and 8 ways to fill the president position. Thus there are (10)(9)(8) = 720 ways to fill these positions. Note this is a different question from asking how many ways there are to choose 3 students from a class of 10 if order does NOT matter. See example 11 below. **Notice that for examples 8, 9 and 10 we used the Fundamental Counting Principle to find the number of permutations (or, combinations in which order matters). This leads us to the formula that the number of ways to choose 3 items from a group of n items, assuming no repeated choices, and assuming that order matters, is:** **(n)(n – 1)(n – 2).** **And the number of ways to choose 4 items from a group of n items is :** **(n)(n – 1)(n – 2)(n – 3).** **In general, the formula for the number of ways to choose k items from a group of n items, with no repeats, and where order matters, is:** **(n)(n – 1)(n – 2)(n – 3)….(n – k + 1).**

Combination of k items from a list of n items, where order is NOT important, can be achieved in a specific number of ways. For example, choosing 3 students from a group of 10, where no student is chosen more than once, can be achieved a specific number of ways. The combinations can be listed in an organized list or on a tree diagram, and then counted. When order is important the technical word to use is Permutation. See above.	11. *How many ways are there to choose 3 students from a group of 10 if no student is chosen more than once and order does NOT matter?* Starting with the Fundamental Counting Principle we have (10)(9)(8) = 720 ways to choose 3 students from a group of 10. But some of these are repeats. Since any 3 items can be arranged in 6 different ways (see example 7) we know that each arrangement in our 720 ways has been duplicated 6 times. So there are in fact 720 ÷ 6 = 120 different ways to choose 3 students from a group of 10. 12. *Find the number of ways that 4 side dishes can be chosen from a list of 5 if there are no repeats and if order does NOT matter, that is find the number of combinations of 4 items from a list of 5.* We could start with the Fundamental Counting Principle, but we have seen that this does not take duplicates into account. So we could go back to an earlier idea and make an organized list of all the ways to choose 4 dishes. Let's call the list of 5 side dishes A, B, C, D, E. The list of combinations of 4 would be ABCD, ABCE, ABDE (these are all the combinations that start AB) ACDE (the only combination that starts AC) We can now search for combinations that start with B, but we should not include any that contain an A, because we already listed all the possibilities with an A. This adds BCDE to the list, making 5 combinations all together. (You should think about why we can't include BACD or BEDC or CABD or CBDE or any other combination to this list.) Note: There is a way to use the Fundamental Counting Principle in example 11, IF we can figure out how many duplicates that would create. The Fundamental Counting Principle tells us that there are (5)(4)(3)(2) = 120 ways to choose 4

	items from a list of 5, if there are no repeats and if the order matters. Now each set of 4 side dishes can be re-ordered in several ways, all duplicates. How many ways can we re-order 4 dishes? Well this is the same as asking how many ways there are to fill 4 slots from a list of 4 items, if order matters. And this would be (4)(3)(2)(1) = 24. So in the 120 ways of selecting 4 dishes from a list of 5, where order matters, each combination of 4 dishes is duplicated 24 times. If we want to eliminate the duplications we compute 120 ÷ 24 = 5. Notice that this answer was also figured out by making a list. We don't have to use formulas to figure out answers.
Factorial a shorthand notation for a multiplication process. For example, 3! (read "3 factorial") means (3)(2)(1) and 4! = (4)(3)(2)(1) n! = (n)(n – 1)(n – 2)….(3)(2)(1)	12. *The Fundamental Counting Principle says that the number of ways to choose 4 items from a group of 6 items is (6)(5)(4)(3). Write this using factorial notation.* 6! is (6)(5)(4)(3)(2)(1). So, (6)(5)(4)(3) = 6! ÷((2)(1)) = 6! ÷ 2! **Note: We can use factorial notation to write a formula for the number of permutations of k items from a group of n items (that is the number of combinations of k items, with no repeats, and where order matters). The Fundamental Counting Principle gives us the answer;** **(n)(n – 1)(n – 2)(n – 3)….(n – k + 1).** **Rewriting this with factorial notation:** **The number of permutations of k items from a group of n items is : n! ÷ (n -k)!**
Network is a diagram, or model, made up of **nodes** and **edges** which represent choices in a particular context. The decisions about which nodes are joined by edges, and by how many edges, are governed by the context of the situation.	13. James can send instructions to his executive secretary by phone, email or fax. The secretary can assign tasks to the office assistants by phone or email. Draw a network to show the number of ways that James's instructions can reach the office assistants.

	 We can trace 6 different routes from J to A. We could also make a list or tree to answer this question, or use the Fundamental Counting Principle.
Note: This unit is designed to help student develop **Combinatorial Reasoning**, that is, to know which questions to ask about a situation involving combinations of choices, such as, “Does order matter?” and to be able to create and use a representation, such as a list or tree or other model, to be able to count the combinations.	