Vocabulary: *Comparing and Scaling*

Concept

Ratios: a comparison between 2 quantities which may or may not have the same unit. Ratios are written in different **formats**. Some of the most often used are, for example, 2 to 3, or 2: 3, or $\frac{2}{3}$.
Note: in elementary school students often compared quantities by subtracting to find a difference. This "difference" way of comparing is appropriate in some situations. As they grow and gain in experience, students develop the ability to recognize situations in which finding a ratio gives an appropriate comparison and when subtracting is called for.

Example

A family has 4 children, 3 girls and a boy. The ratio of *girls to boys* is 3 to 1 or 3: 1 or $\frac{3}{1}$. If another boy is born to the family the ratio becomes 3: 2, which can also be written as $\frac{3}{2}$. (The order is important. The ratio of *boys to girls* has become 2:3 or $\frac{2}{3}$.)

Another example:
A family buys a new hybrid car and notices that they get *800 miles per 16 gallon* fillup, or 800: 16 or 50:1. (Notice that this time the ratio is made of quantities which have different units, miles and gallons).

Example showing the difference between thinking of a **comparison as a difference** and thinking of a **comparison as a ratio**.:
Aaron buys two new fish for his aquarium. The yellow striped fish is 3 inches long when he buys it, the blue fish is 6 inches long. After a month he notices that the yellow fish is 5 inches long and the blue fish is 8 inches long. *Compare* how these fish are growing.
There are several ways to make correct comparisons here. The *"subtracting*" way to make a comparison is to say that both fish have grown by an equal amount, 2 inches. (You could correctly write this as 2 inches:1 month.) However, this "subtracting" way of thinking of the growth has not made use of the original size in the comparison. The yellow fish's *new size to original size* = 5:3 or $\frac{5}{3}$, while the blue fish's *new size to original size* = 8:6 or $\frac{8}{6}$. Since $\frac{5}{3}$ is more than $\frac{8}{6}$, this way of making the comparison makes the yellow fish's growth more impressive. (Or yellow fish's *growth to original size* = 2:3, while blue fish's *growth to original size* = 2:5.)

Connecting Ratio Statements to Fraction Statements of Comparison:

While ratios can be written in a fraction format they are not always a part to whole relationship, while fractions always are. Ratios may be a part to part comparison or a part to whole comparison. . As well as developing the ability to move between ratio and fraction representations of any situation, students need to develop the confidence to choose the form of representation that best describes the situation.

There are 9 boys and 12 girls in a class. The ratio of *boys to girls* is 9:12 or $\frac{9}{12}$. We could correctly rewrite this ratio as 3:4 or $\frac{3}{4}$. And we could correctly say that the number of boys is $\frac{3}{4}$ of the number of girls. But we can not say that the boys are $\frac{3}{4}$ of the class. In this last statement the "whole" is the entire class. We would have to return to the original statement 9 boys to 12 girls to make a statement comparing the boys to the whole class, 9 boys in a class of 21, or the boys are $\frac{9}{21}$ or $\frac{3}{7}$ of the class.

Another situation:
An inheritance is to be split among 4 children; the oldest gets 4 parts, the next 3 parts, the next 2 parts and the youngest 1 part. Comparing the youngest child's inheritance to the oldest, we can say that the ratio is 1:4 or we can say that the youngest gets $\frac{1}{4}$ as much as the oldest. But if we want to say anything about the whole inheritance we have to say that the youngest child's inheritance is 1 part out of the total of 10 parts, or $\frac{1}{10}$ of the total, while the oldest gets $\frac{4}{10}$ of the total. It is important to specify what we are comparing and what the "whole" is.

Connecting Ratio Statements to Percentages:
Percentage statements are always about finding a part "of" some whole; x% means x parts out of a total of 100. Ratio statements can be rewritten in percentage form as long as care is taken to identify what the "whole" is. Students need to learn to discriminate between representations and to choose the form of representation that best describes the situation.

If a punch recipe calls for only 2 ingredients, 2 cups water and 5 cups juice, then the ratio of water to juice is 2:5 or $\frac{2}{5}$. We can say that the amount of water is $\frac{2}{7}$ *of the total* punch, or about 29% of the punch is water.

Suppose in a recipe for a snack mix we need 1 *package* of candy and 4 cups cereal. We could say the ratio of candy to cereal is 1 package: 4 cups. We know that $\frac{1}{4}$ could be written as a percentage, 25%, but it makes no sense to say that the amount of candy is $\frac{1}{4}$ or 25% of the amount of the cereal because the units of measurement are not the same. Nor does it make sense to add to get a total, and say that the ratio of candy to total mix is 1:5, since the units are different. Candy is not 20% of the mix.

Another example:
If it takes 8 hours to complete a 400 mile car journey then the ratio of *time to distance* is 8 hours: 400 miles, or more simply 1 hour: 50 miles. It makes less sense to think of this as a fraction or percentage. The time is not 2% of the distance, since they are not measured in the same units. (Though one could say that the *number* of hours is 2% of the *number* of miles.) Again, adding the parts of the ratio is not appropriate since adding miles and hours would not make sense.

Scaling Ratios: is a strategy for making comparisons between two ratios. It calls for the same mathematical thinking used to find **common denominators or common numerators** in work with equivalent fractions.

A recipe calls for 4 ounces of chocolate and 3 eggs, that is 4:3 or $\frac{4}{3}$. If we want to **scale** this recipe up, we can write 4:3 = 8:6 (**multiplying the numerator and denominator by 2)** or 12:9 (multiplying the numerator and denominator by 3) etc. Likewise we can scale down ratios.

If a recipe calls for 4 ounces chocolate and 6 tablespoons of sugar and we want to use up all of a 10 ounce bar of chocolate, how much sugar do we need? The question becomes, Chocolate: sugar = 4 ounces: 6 Tablespoons = 10 ounces: ? Tablespoons, *or* $\frac{4}{6} = \frac{10}{x}$ *?* We can scale $\frac{4}{6}$ down to $\frac{2}{3}$ (dividing numerator and denominator by 2 to rename the fraction) and then scale this up to $\frac{10}{15}$ (multiplying numerator and denominator by 5 to get the required numerator, 10). So we need 15 tablespoons of sugar.

Unit rates: are ratio statements of one quantity *per one unit* of the other quantity. Any given ratio can be rewritten as 2 different unit rate statements, though one of these may make more sense in the given context. **Rate tables** are a way of setting up and rewriting rates to answer a specific question. This connects to work done on rates in *Variables and Patterns*.

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Here is a situation where **unit rates** can be calculated in more than one way, each of which helps interpret the situation:
Sasche goes 6 miles in 20 minutes on leg 1 of his bike ride. On leg 2 he goes 8 miles in 24 minutes. On which leg is he riding faster?

If we divide 20 by 6 we get 3.333; as a ratio this means *20 minutes: 6 miles = 3.333 minutes: 1 mile, or 3.333 minutes per mile*, a **unit rate**. If we divide 24 by 8 we get 3; as a ratio this means 24 minutes: 8 miles = 3 minutes: 1 mile, or 3 minutes per mile, a **unit rate**. He takes a shorter time for each mile on the second leg.

OR
If we divide 6 by 20 we get 0.3; as a ratio this means *6 miles: 20 minutes = 0.3 miles: 1 minute, or 0.3 miles per minute, a* **unit rate**. Similarly 8 miles: 24 minutes = 0.333 miles: 1 minute, or 0.333 miles per minute, a **unit rate**. He rides further in each minute on the second leg.

You can also compare these rates by making **rate tables**. The rate table for leg 1:

Distance in miles	**Time in minutes**
6	20
$\frac{6}{6}$ = 1 (scaling down initial information by dividing by 6)	$\frac{20}{6}$ = 3.333 (This gives a unit rate, 3.333 minutes per mile.)
$\frac{6}{20}$ = 0.3 (scaling down the initial information by dividing by 20)	$\frac{20}{20}$ = 1 (This gives a unit rate, 0.3 miles per minute)
6 x 4 = 24 (scaling up original information by multiplying by 4)	20 x 4 = 80 (80 minutes for 24 miles)

A rate table for leg 2:

Distance in miles	**Time in minutes**	
8	24	
$\frac{8}{8}$ = 1 (scaling down initial information by dividing by 8)	$\frac{24}{8}$ = 3 (This gives the unit rate, 3 minutes per mile.)	
$\frac{8}{24}$ = 0.333 (scaling down the initial information by dividing by 24)	$\frac{24}{24}$ = 1 (This gives the unit rate, 0.333 miles per minute)	
8 x 3 = 24 (scaling up original information by multiplying by 3)	24 x 3= 72 ***(This allows a comparison with leg 1 since both distances have been scaled to 24 miles. Leg 2 is faster, at 72 minutes for 24 miles, while leg 1 was 80 minutes for the same distance.)***	

Proportions: are statements of equality between two ratios, one of which is usually incomplete.

Proportional Reasoning: is the ability to find the comparisons within a situation, recognize that forming ratios is the appropriate way to make the comparisons, and then use strategies such scale factor, or scaling up or down, or rate tables, or finding equivalent fractions to find a missing part of a proportion.

Suppose the EPA wishes to study the salmon population of a lake, but does not know how many salmon there are in the lake in all. 20 salmon are caught, tagged and released. Now we know that there are only 20 tagged salmon, but we do not know the total. The ratio of tagged to total is 20:S Later 36 salmon are caught from the same lake, and only 8 of them have tags. Assuming that this sample of 36 salmon proportionally represents the total population of salmon in the lake we can write:
8 tagged:36 in total sample = 20 tagged:x total population.
$\frac{8}{36} = \frac{20}{S}$.
This is a proportion with one part missing. It can be solved many different ways, all using ideas with which students are familiar from prior units.

Scale factor:
We can ask what scale factor would it take to *scale up 8 tagged salmon to 20 tagged salmon*. This is a familiar idea from Stretching and Shrinking. In this case the *scale factor is* $\frac{20}{8}$ or 2.5. Then we can scale up the sample by the same scale factor, to get:
$\frac{8}{36} = \frac{8\times 2.5}{36\times 2.5} = \frac{20}{90}$. There are 90 salmon in the lake.

Common numerator:
This relies on understanding how to rewrite fractions as equivalent fractions.
$\frac{8}{36} = \frac{20}{S}$
$\frac{8\times 5}{36\times 5} = \frac{20\times 2}{S\times 2}$
$\frac{40}{180} = \frac{40}{2S}$
Therefore, 2S = 180, so S = 90.

Common denominator
This variation is useful when the missing quantity

is in the numerator, and both denominators are given.

For example, $\frac{3}{10} = \frac{S}{12}$.

We can rewrite the two fractions as

$$\frac{3\times6}{10\times6} = \frac{S\times5}{12\times5}$$

$$\frac{18}{60} = \frac{5S}{60}$$

Therefore, 5S = 18, so s = 3.6.

Rate table:

Solve: $\frac{3}{10} = \frac{S}{12}$

3	$\frac{3}{10}$	$(\frac{3}{10})(12)$	S
10	1	12	12

So, S = $(\frac{3}{10})(12)$ = 3.6.

OR

3	36	……	S	10S
10	120	…….	12	120

So, 10S = 36, so S = 3.6.

Fact family:

Solve:

$\frac{3}{10} = \frac{S}{12}$

0.3 = S ÷ 12

So, S = 0.3 x 12 = 3.6.

Generalized methods for solving proportions: There are several general methods that always work. The best of these have the advantage of being so strongly connected to basic ideas about equivalence of fractions or scale or rate that they make sense to students and can be recreated as needed. *Cross-multiplying*, a traditionally taught method that does *not* have the quality of making sense to students, is often misused by students and is therefore not taught in this unit..

Scale factor:

$\frac{3}{7} = \frac{x}{343}$

Scale factor needed to increase denominator 7 to make denominator 343 is $\frac{343}{7}$. Therefore, this is the scale factor that should be applied to 3 to get x.

In general,

$\frac{a}{b} = \frac{x}{d}$

Scale factor needed to increase denominator b to make denominator d is d/b. Therefore, this is the scale factor that should be applied to a to get x.

X = $(\frac{d}{b})$(a).

Fact family:

$\frac{3}{7} = \frac{x}{343}$

$0.42857 = x \div 343$
Therefore, $x = 343(0.42857)$
In general,
$\frac{a}{b} = \frac{x}{d}$
$(\frac{a}{b}) = x \div d$
Therefore, $x = (\frac{a}{b})(d)$

Unit rate:
$\frac{3}{7} = \frac{x}{343}$
$\frac{\frac{3}{7}}{1} = \frac{x}{343}$, now scaling up the first denominator to make them both 343,
$(343)(\frac{3}{7}) = x$.
In general:
$\frac{a}{b} = \frac{x}{d}$
$(\frac{a}{b})/1 = \frac{x}{d}$
$d(\frac{a}{b}) = x$