# Vocabulary: *Covering and Surrounding*
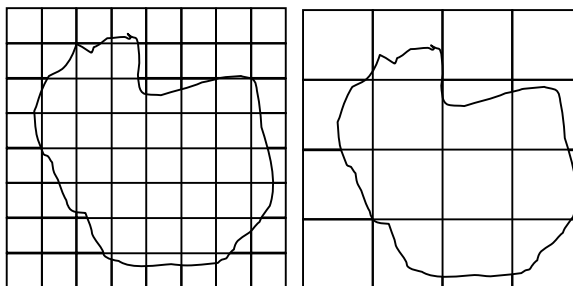
## Concept
## Measurement:

The measurement process involves several key elements.

- Identify the attribute that can be measured. In this unit the focus will be **perimeter and area**.

- Select an **appropriate unit**. In the context of *Covering and Surrounding* these units will be inches, feet, yards, miles or centimeters, meters, kilometers (for perimeter), and square inches, square feet etc. (for area). Choosing **smaller units of measure permits more accuracy** in measurement..

- Use the unit chosen to "match" against the attribute; for example, inch lengths can be laid around a shape's outline, and inch squares can be used to cover a shape.

Note: All measures are approximate. One can fine-tune measurements to get a degree of precision in a particular situation, but no matter how precise the instruments with which a measurement is made, *error will always exist*. This "error" is not a "mistake," but an inevitable part of a process which depends on concrete observations. One can develop "rules" which can be applied to given measurements of particular figures. These rules operate independent of human observation, but they operate on given measurements, which may themselves have inbuilt error.

## Example
Estimate the **area** of the irregular shape shown below. (Same shape is shown on each grid.)



### Estimate A.
If we use the grid with small square units we can count 19 whole squares and 26 partial squares. Some of the partial squares are over a half square, some are almost a whole square, some are less than half a square and some are very small. If we estimate what fraction of a square each of these partial squares might be, and add the results we arrive at an estimate of approximately 12 whole squares. The total area is, therefore, 19 + 12 = 31 square units.
### Estimate B.
If we use the grid with the larger squares we see we have 3 whole squares and 10 partial squares. The partial squares cover approximately 5 whole squares. This makes a total of 8 whole squares.
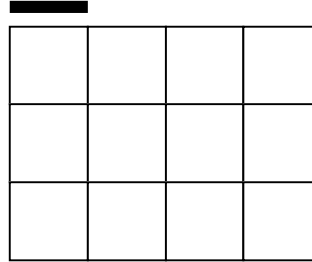
The answers are different, not because one is correct and the other is wrong, but because we used different units to measure the area. Since each large square is actually 4 small squares estimate B should be multiplied by 4 before comparing it to estimate A. But this would make estimate B equal to 32 small squares on the small grid, not 31 small squares. Why the difference? The difference is caused in part by the fact that both answers are just approximations. However, there is likely to be **less inaccuracy** in estimate A, because, with the **smaller size of unit**, we could confidently count more of the area covered with whole small squares, leaving less area to be estimated.

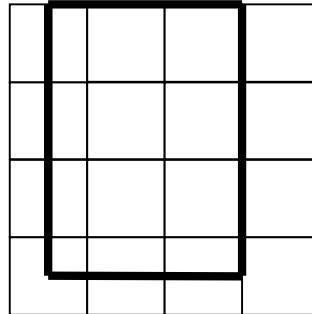**Strategies for finding perimeter of a rectangle:**

- Build a rectangle to fit the given dimensions with square tiles (if a whole number of tiles will work) and count the "edge" units. (Students may actually have to physically lay a concrete unit length against the edges to avoid missing any of the edge units. Students may also notice that they only have to find the lengths of two adjoining sides, because the rectangle is made of two pairs of identical sides)
- Cover the rectangle with a grid (square inch or square centimeter). Line up the grid lines with two adjoining sides of the rectangle. Count the unit lengths and partial unit lengths on the grid for these two adjoining sides. Use this to find the lengths of the other two sides of the rectangle, and add all four side lengths.
- Use a ruler to measure two adjoining sides of the rectangle, and add all four side lengths.
- Use the "rule" that **perimeter = 2lengths + 2 widths or perimeter = 2(length + width).**

*1. What is the perimeter of a rectangle with length 4 units and width 3 units?*

The rectangle below has the correct dimensions. The heavy line segment represents 1 linear unit. We can lay this line segment 4 times along the top of the rectangle (and 4 times along the bottom). We can also lay the line segment 3 times along each side. The **perimeter** is 4 + 3 + 4 + 3 = 14 units.
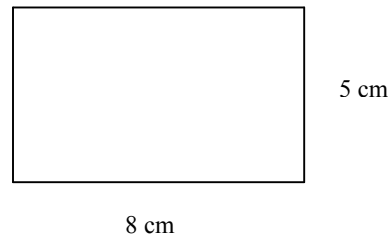


*2. What is the perimeter of the rectangle shown on the grid below?*



The grid is aligned with two sides of the rectangle, so we can see that that length of the top is approximately 2.5 units, and the length of the side is approximately 3.5 units. Therefore, the **perimeter** is 2.5 + 2.5 + 3.5 + 3.5 = 12 units.
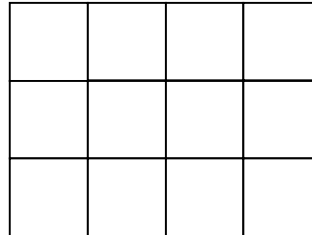
*3. What is the perimeter of the rectangle shown below?*



5 cm

8 cm

Using the **rule**,

**perimeter** = 8 + 5 + 8 + 5 or 2 x 8 + 2 x 5 or 2(8 + 5) = 26 units. (Notice we do not need labels on the other sides because we know that opposite sides are equal in length. )
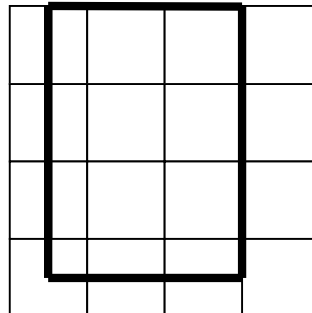
**Strategies for Finding Area of Rectangle**:
- Build a rectangle with given dimensions using square tiles, and count the number of tiles.  Note: students quickly discover that they have to count the number of squares in one row and multiply by the number of rows.
- Use a grid.  Lay the grid over the rectangle so that the grid lines line up with two adjacent sides of the rectangle.  Count the number of squares or partial squares that cover the rectangle.
- Use the rule:
  **area of a rectangle = length x width, or, area of rectangle = base x height.**

1. *What is the area of a rectangle with length 4 units and width 3 units?*



If we construct this rectangle we have to make 3 rows of 4, covering 12 square units. **Area** = 12 square units.

2. *What is the area of the rectangle shown on the grid below?*



Counting the square units covered we see 6 whole squares and 6 partial squares.  Estimating the fraction of a square that each of the 6 partial squares covers we might have 2.75 whole squares.  Thus the estimated **area** of the rectangle is 6 + 2.75 = 8.75 square units.

3**. If** the exact dimensions of the rectangle shown on the above grid are length 2.5 units and width 3.5 units, then the area, using the **rule**, is
**Area = 3.5 x 2.5 = 8.75** square units.

**Connections between Perimeter and Area of a Rectangle:**
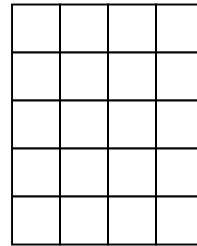
A given perimeter can enclose many different

*1. How many different rectangles can be made with the perimeter 18 units?*
Assuming we can only use whole numbers for the dimensions, we could have a length of 1 and a
width of 8 units, making a **half perimeter of 9**

rectangular areas.  It is not true that rectangles with equal perimeters must have equal areas.

A given area can be surrounded by many different perimeters.  It is not true that rectangles with equal areas must have equal perimeters.

width of 8 units, making a **half perimeter of 9 units**, or a whole perimeter of 18 units; or we could have a length of 2 and a width of 7 (length + width must be 9 units)); or a length of 3 and a width of 6; or a length of 4 and a width of 5.  (If we also list a length of 5 and a width of 4 this rectangle will be congruent to another rectangle already listed, though the orientation has changed because the length and width have switched.)  Thus, we have 4 distinct rectangles, all with perimeter 18 units.  (If we are permitted to use fractions for the length and width then we could have 0.5 and 8.5, or 1.2 and 7.8 etc.  There would be an infinite number of combinations that add to 9 units.)  Two examples are shown below.



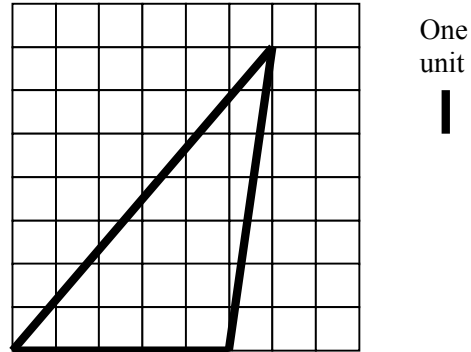Perimeters are both 18 units,
Areas are different.

*2.  How many rectangles are there with the area 24 square units?*
If the area is 24 square units then length x width = 24, so we are looking for a pair of numbers whose product is 24.  The length could be 1 and the width could be 24; or the length could be 2 and the width 12; or the length could be 3 and the width 8; or the length could be 4 and the width 6.  If we are permitted to use only whole numbers for the dimensions then there are 4 distinct rectangles with area 24 square units.  (As above, if we are permitted to use fractions for the dimensions then there are an infinite number of combinations.)

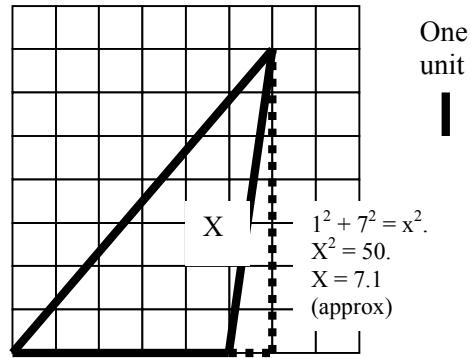**Strategies for Finding Perimeter of Triangle:**

- Lay unit lengths or a ruler along the edges of the triangle and add the lengths. Since the triangle may have three different lengths for sides every side must be measured. Using a grid will permit the measure of only one side (two sides if the triangle is right angled). In general, **p = a + b + c.**

- Surround the triangle with a string loop. Open the string loop and use a ruler to measure the length.

- If the triangle is drawn on a grid then using the Pythagorean Theorem may be applicable. (Students do not meet the Pythagorean Theorem until 8th grade.)

1. *Find the perimeter of the triangle shown below.*



One unit

The shortest side is 5 units. We can see this because it lines up with the grid. The medium side is a little more than 7 units long. We can see that this line is almost vertical so we can compare its length to a nearby grid line. (Students will often say that its length IS 7 units because it crosses 7 squares, but the slant distance across a square is always longer than the length of the edge of a square.) The longest side is hardest to estimate. We could lay the length of 1 unit along this edge multiple times. It seems to be about 9 units long. So the **estimated perimeter is 5 + 7 + 9 = 21 units.**

(The slant lengths in the above triangle are difficult to estimate. When students are in grade 8 they are introduced to the Pythagorean Theorem and can calculate these slant lengths with great accuracy. Shown below is the calculation for the medium side. Using the Pythagorean Theorem the perimeter would be approximately, 5 + 7.1 + 9.2 = 21.3 units.)
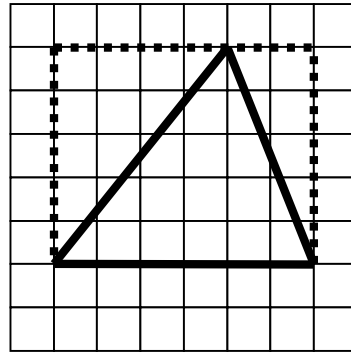
One unit

$1^2 + 7^2 = x^2.$
$X^2 = 50.$
$X = 7.1$
(approx)

X

**Strategies for Finding Area of Triangle:**

- Lay a grid over the triangle and count the number of squares and partial squares covered by the triangle.

- Divide the triangle into pieces that may more easily be measured (usually rectangles and right triangles). Add the partial areas.

- Surround the triangle with a rectangle, so that the base of the rectangle coincides with a side of the triangle, and the height of the rectangle coincides with the height of the triangle. Find the areas of the "extra" pieces of the rectangle (which can be done by counting squares and partial squares) and subtract these from the area of the rectangle.

- Use a rule. Since any triangle can be thought of as half of a rectangle with the same **base** and **height**, we can take advantage of the rule for the area of a rectangle. Thus,

**Area of triangle = ($\frac{1}{2}$)(base x height)**

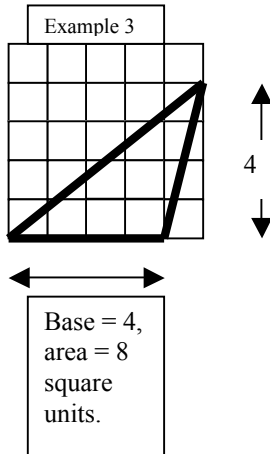1. *Estimate or calculate the area of the triangle shown below.*
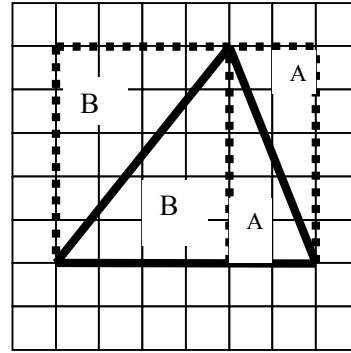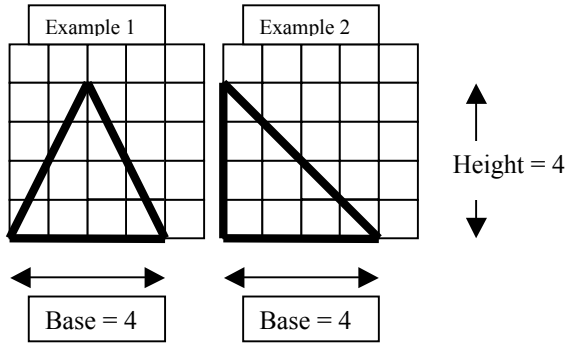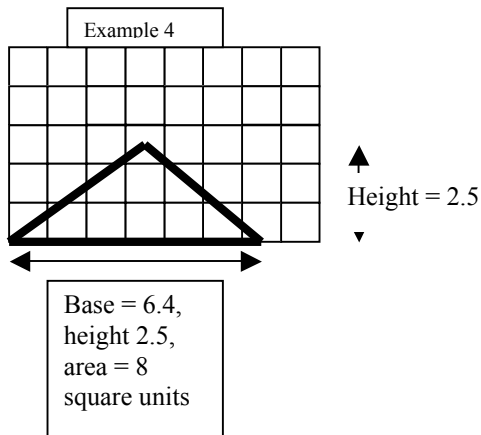


We could estimate this area by
- Counting whole squares and estimating the sizes of the partial squares, or

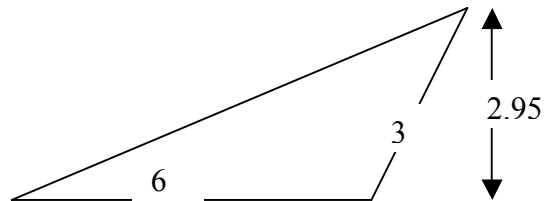We could calculate the area of the triangle
- By surrounding the triangle with a rectangle (shown as broken lines). The rectangle is made of two right triangles and the required triangle. If we subtract the areas of the right triangles we will have the required area. Or
- By surrounding the triangle with a rectangle as above and noting that the right triangles can be rearranged to make the same area as the given triangle. (See below) The triangles labeled "A" are identical. The triangles labeled "B" are identical. Thus, the required triangle is half of the 5 by 6 rectangle. **Area of triangle = ($\frac{1}{2}$)(5 x 6) = 15** square units.

Example 1    Example 2

Height = 4

Base = 4    Base = 4



Example 3

4

Base = 4,
area = 8
square
units.

Examples 3
and 4 are the
same
triangle, with
different
orientation.
If you choose
a different
base you
must choose
a height to
correspond to
the chosen
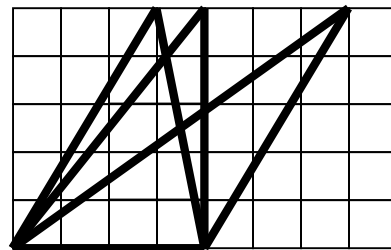base.

2. *Calculate the area of the triangle shown below.*



2.95

3

6

The only **perpendicular height** given is 2.95
units, and this rises from **the base of 6** units to
the opposite vertex. (The "3" is not needed for
this calculation. It is a **slant height**.)
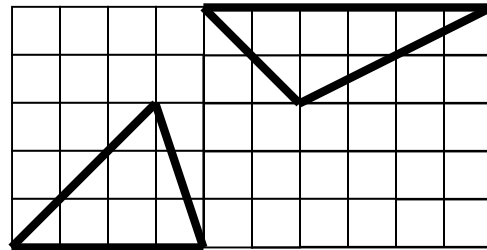**Area of triangle = ($\frac{1}{2}$)(6 x 2.95)** = 8.85 square
units.

3. *If two triangles have the same base and height
must they have the same area?*

Shown below are three triangles, an acute
triangle, a right triangle, and an obtuse triangle, all
sitting on the same base of 4 units. They each
have the same height of 5 units. The area for
each is ($\frac{1}{2}$)(4 x 5) = 10 square units.

Example 4



Height = 2.5

Base = 6.4,
height 2.5,
area = 8
square units

4. *If two triangles have the same area must they have the same base and height?*
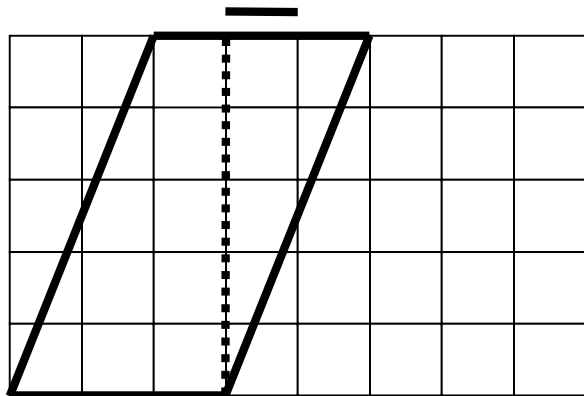The triangles shown below have, respectively, base 4 and height 3, and base 6 and height 2. (The base does not have to be on the "bottom" of the triangle, nor does it have to be oriented horizontally.) Therefore, respective areas are $(\frac{1}{2})(4 \times 3) = 6$ square units, and $(\frac{1}{2})(6 \times 2) = 6$ square units. The areas are the same but the bases and heights (and triangles) are quite different.



**Strategies for Finding Perimeter of Parallelograms:**

- Lay unit lengths or a ruler along the edges of the parallelogram and add the lengths. Since the parallelogram has two pairs of equal sides only two need to be measured. (Using a grid will permit the measure of only one side, unless the parallelogram is a rectangle.)

- Use a rule. P = 2a + 2b, assuming a and b, the lengths of the sides, are given.

**Strategies for Finding Area of Parallelograms:**

- Cover with a grid and count the squares and partial squares covered by the parallelogram.

- Divide the parallelogram into easier shapes, such as rectangles and triangles.

1. *Find the perimeter of the parallelogram shown below.*



As with a triangle it is going to be difficult to estimate the perimeter of the slant edges. We have two sides lined up with the grid, each 3 units long. The other sides can be estimated using a unit length as a comparison. (Later students can calculate these lengths using Pythagorean Theorem.) The **perimeter** is approximately 3 + 3 + 5.5 + 5.5 = 17 units.

- Use a rule: Since every parallelogram can be created by combining two congruent copies of a triangle the area of the parallelogram is therefore twice the area of the triangle with the same **base** and **height**. That is,
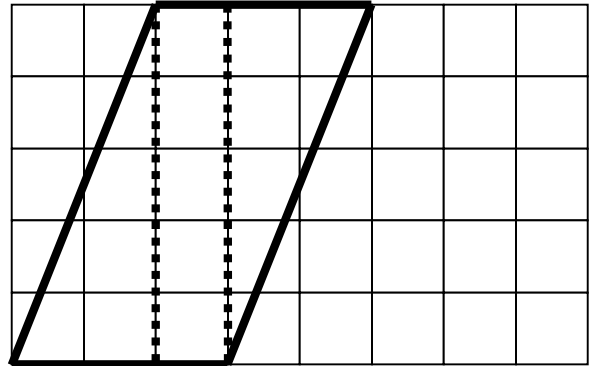
**Area of parallelogram = 2(triangle area) = 2($\frac{1}{2}$)(base x height) = base x height.**

Note: the **base** of the parallelogram is a side of the parallelogram, but the **height** is not a side. The choice of a particular side for a base fixes the corresponding height as the perpendicular distance from the base to the opposite side.
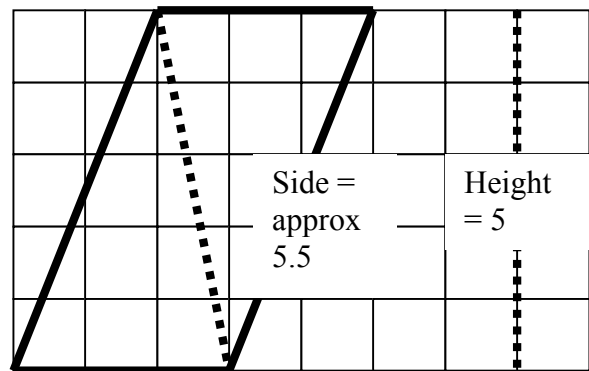
2. *Find the area of the above parallelogram.*

If we cut the parallelogram along the vertical line, shown as a broken line above, and reassemble the two parts we can make a rectangle with base 3 and height 5. So the area of the parallelogram is 15 square units.

Or we can cut the parallelogram into a rectangle and two right triangles, as below.



Or we can divide the parallelogram into two parts by cutting along either diagonal (see below). This creates two congruent triangles and the area of each triangle is ($\frac{1}{2}$)(base x height). So the area of the parallelogram is twice this, that is Area = 2($\frac{1}{2}$)(3 x 5) = 15 square units.

u



| Side = approx 5.5 | Height = 5 |

Base = 3

Or we can use the rule, Area = base x height = 3 x 5 = 15 square units.

**Strategies for Finding Circumference of Circles:**

As with rectangles the perimeter or circumference depends on the dimensions of the shape. But with a circle there is only one dimension, the **diameter** (or **radius**, which is a half diameter). The diameter is the length of a segment that passes through the center and has ends on the circumference.

- We can wrap a string loop around the circle and then use a ruler to measure the length of the string loop.

- We can use a string which is the same length as the diameter, and estimate how many diameters will surround the circle.

- We can use a **rule** about the number of diameters that wrap around a circle.
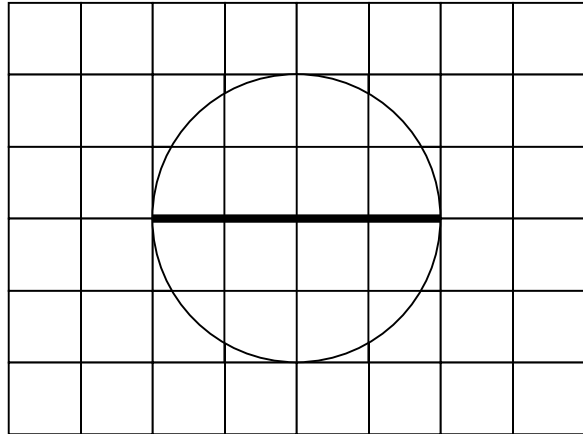
  **C = pi times diameter, or 2pi times radius.**

  **(The symbol π, written also as "pi" is used to stand for the number of diameters that wrap around a circle. It is approximately 3.14, but not exactly any fraction or terminating or repeating decimal.)**

**Strategies for Finding Area of Circles:**

As with a rectangle the area depends on the dimensions of the shape. But, the circle has only one dimension, diameter or radius.

1. *Find the circumference of the circle shown below.*



Students could
- measure the diameter (4 units) and cut a piece of string or paper the same length and place this repeatedly around the circle. They will find that the circumference is slightly over 3 times the length of the diameter, or about 12.5 units long.
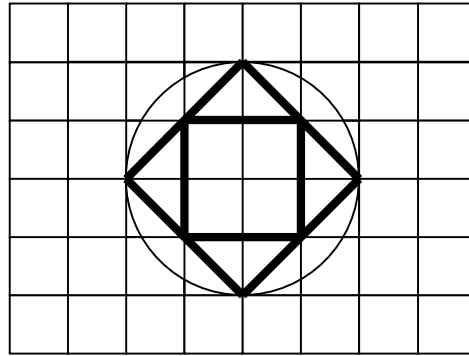
  Note: This relationship between diameter and circumference is true, no matter the size of the circle.

- Use the rule, **Circumference = π times diameter = 3.14(4) = 12.56 units.**

1. *Estimate the area of the circle shown below.*

- We can lay a grid over the circle and count squares and partial squares.

- We can lay a grid over the circle, and divide the enclosed area into easier shapes such as rectangles and triangles, and then estimate any leftover partial squares.

- We can create a "**radius square**", and lay this over the circle to estimate how many "radius squares" will fit over the area.

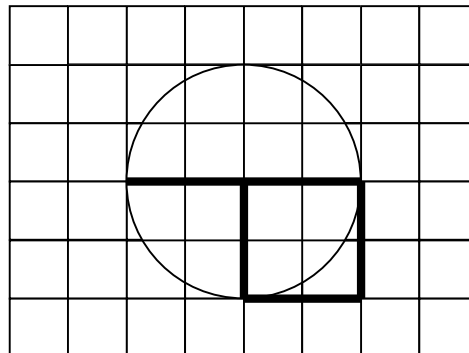- We can apply a rule,

  **Area = pi x radius square**



The area can be divided up into a square and triangles, as shown above, each of which has an area which is easy to compute. Then there are partial square units which will have to be estimated. The estimated area

$= 4 + (\frac{1}{2})(2 \times 1) + (\frac{1}{2})(2 \times 1) + (\frac{1}{2})(2 \times 1) + (\frac{1}{2})(2 \times 1) +$ area of 4 curved sectors

$= 8 +$ approximately 4 more square units =

$= 12$ square units (approx)

2. *What is the area of a radius square, and how many radius squares will cover this circle?*



The square on the radius is 4 square units. Four of these squares will be too large to cover the circle. So 16 square units is too large an estimate. On the other hand three radius squares is too small an estimate. So the area of the circle is between 12 and 16 square units.

3. *Calculate the area of the circle shown above.*
Using the **rule A = π (radius)$^2$** = $3.14(2)^2$ = 12.56 square units (approx).

Note: the area and circumference of the above circle each had the same number of units, 12.56. This only happened because the diameter (needed in the circumference calculation) is 4, as is the radius$^2$, needed in the area calculation. If the diameter had been 6 then the Circumference would be C = 3.14(6) = 18.84 units (approx). The area for the same circle would be A = 3.14(3)$^2$ = 28.26 square units (approx.)

**Maximizing or Minimizing a Measurement Given One Fixed Attribute**:
Some students hold the misconception that if you know the area of a shape that you can find the perimeter. Yet, for any given area measurement you can make many different shapes with that area, all of which could have varying perimeters.

- If we fix the perimeter in rectangle, we can find a minimum or maximum area.
- If we fix the Area in rectangle we can find a min/max perimeter
- If we fix the perimeter of triangle we can find a min/max area.
- If we fix area of triangle the perimeter varies.
- If we fix the Circumference of a circle there is ONLY 1 circle possible.
- If we fix the Area of a circle there is ONLY 1 circle possible.
- If we fix a perimeter we can find a shape that will cover the maximum possible area surrounded by this perimeter.

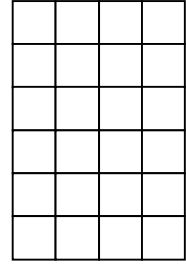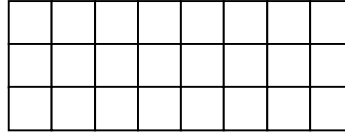1. *What is the maximum area for a rectangle whose perimeter is 18 units?*

As we saw above there are many rectangles with a perimeter of 18.

| Length | Width | Perimeter | Area |
|--------|-------|-----------|------|
| 1 | 8 | 18 | 8 |
| 2 | 7 | 18 | 14 |
| 3 | 6 | 18 | 18 |
| 4 | 5 | 18 | 20 |
| 5 | 4 | 18 | 20 |
| Etc. | | | |

If we are permitted to use fractions there will be even more possible rectangles. In the above table the maximum area appears to be 20 square units. However, if we use 4.5 for both length and width then the perimeter will still be 2(4.5 + 4.5) = 18 units, but the area will be 4.5$^2$ = 20.25 square inches. In general, the more "square" the rectangle (that is, the closer the ratio of sides is to 1:1) then the larger the area enclosed by the given perimeter.

2. *What are the maximum and minimum perimeters of a rectangle whose area is 24 square units (using only whole numbers for the dimensions).*

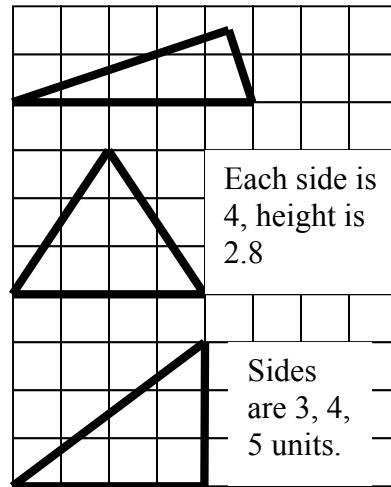Here are some rectangles with the area 24 square units.

As you can see the perimeter varies, because the dimensions vary.  The table below organizes this information.

| Length | Width | Area | Perimeter |
|--------|-------|------|-----------|
| 1 | 24 | 24 | 50 |
| 2 | 12 | 24 | 28 |
| 3 | 8 | 24 | 22 |
| 4 | 6 | 24 | 20 |
| 6 | 4 | 24 | 20 |
| etc | | | |

The maximum perimeter is 50; the minimum perimeter is 20. (If we are permitted to use fractions for length and width there would be even more possibilities.  It seems as if the minimum perimeter would occur at length 5 and width $\frac{24}{5}$ .

This would give a perimeter of 19 $\frac{3}{5}$.  Actually, this is not quite the exact perimeter, but we would need some knowledge of calculus to arrive at a more accurate answer. There would be no maximum since we could choose any fraction less than 1 for the length, and choose a width to correspond.  This would give a perimeter greater than 50. )

3. *What is the maximum area that can be enclosed in a triangle with perimeter 12 units?*
There are many different shaped triangles (right, acute, obtuse, isosceles etc.).  I
Shown below are some triangles with perimeter 12 units.

Each side is 4, height is 2.8
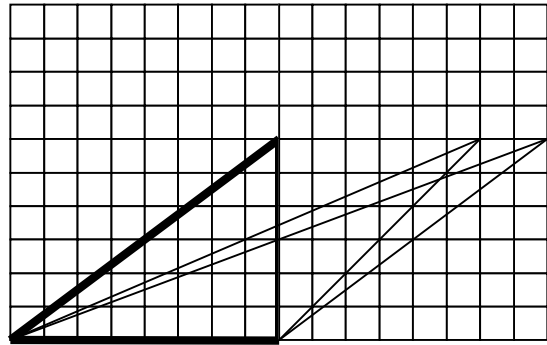
Sides are 3, 4, 5 units.

It appears that the equilateral triangle encloses the largest area. Area = $(\frac{1}{2})(4)(2.8)$ = 5.6 square units.

4. *What are some triangles that enclose an area of 24 square units?  How do the perimeters vary?*
If the area of a triangle is 24 square units then we can have many different base/height combinations.  Some of these are:

| Base | Height | Area |
|------|--------|------|
| 1 | 48 | 24 |
| 2 | 24 | 24 |
| 3 | 16 | 24 |
| 4 | 12 | 24 |
| 6 | 8 | 24 |
| 8 | 6 | 24 |

Suppose we investigate triangles with base 8 and height 6, we discover that we can draw many different triangles and the perimeters are all different.

It seems that the longer and narrower the triangle the greater the perimeter. The triangles get longer and narrower as one of the base angles gets more and more obtuse. If we apply this to "narrowest" of the base-height combinations in the table, base 1 and height 48, we can make many obtuse triangles with base 1 and height 48. By making the base even smaller or the base angle even more obtuse we can increase the perimeter.