Vocabulary: ***Growing, Growing, Growing.***

<table>
<tr><th>Concept</th><th>Example</th></tr>
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Exponential Growth: An exponential pattern of change can often be recognized in a verbal description of a situation or in the pattern of change in a table of (*x*, *y*) values. In general, one variable, y, is said to be **growing exponentially** with respect to another variable, x, if, for each increment of one unit in x, y increases by **multiplying** the last value of y by a constant factor.

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1. Which of these tables illustrates a linear growth pattern for y, and which an exponential growth pattern?

a

<table>
<tr><td>X</td><td>Y</td></tr>
<tr><td>0</td><td>1</td></tr>
<tr><td>1</td><td>3</td></tr>
<tr><td>2</td><td>9</td></tr>
<tr><td>3</td><td>27</td></tr>
<tr><td>4</td><td>81</td></tr>
</table>

b

<table>
<tr><td>x</td><td>y</td></tr>
<tr><td>0</td><td>5</td></tr>
<tr><td>1</td><td>8</td></tr>
<tr><td>2</td><td>11</td></tr>
<tr><td>3</td><td>14</td></tr>
<tr><td>4</td><td>17</td></tr>
</table>

c

<table>
<tr><td>x</td><td>y</td></tr>
<tr><td>0</td><td>1</td></tr>
<tr><td>1</td><td>3</td></tr>
<tr><td>2</td><td>6</td></tr>
<tr><td>3</td><td>10</td></tr>
<tr><td>4</td><td>15</td></tr>
</table>

d

<table>
<tr><td>x</td><td>y</td></tr>
<tr><td>0</td><td>100</td></tr>
<tr><td>1</td><td>150</td></tr>
<tr><td>2</td><td>225</td></tr>
<tr><td>3</td><td>337.5</td></tr>
<tr><td>4</td><td>506.25</td></tr>
</table>

a. The y-values change by a factor of 3 as the x values change by increments of 1. Thus, 1 x 3 = 3, 3 x 3 = 9, 9 x 3 = 27 etc. This is an **exponential** or *multiplicative* pattern of growth.

b. The y-values change by increasing by 3 each time the x values change by and increment of 1. Thus, 5 + 3 = 8, 8 + 3 = 11, 11 + 3 = 14 etc. This is a **linear** or *additive* growth pattern. (See *Moving Straight Ahead* unit for more information on linearity.)

c. Shows neither exponential nor linear pattern of growth (though there is a pattern).

d. This is an **exponential** pattern of growth. The multiplicative factor is 1.5. Thus 100 x 1.5 = 150, 150 x 1.5 = 225 etc.

Note: It is important to distinguish between a constant growth *factor* (multiplicative), as just illustrated in an exponential pattern, and the constant *additive* pattern in linear relationships.

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Growth Factor

The constant multiplicative factor referred to for Exponential Growth (above) is called the **growth factor.** This constant factor can also be obtained by dividing each successive *y*-value by the previous *y*-value.

2. *What is the* ***growth factor*** *in the two tables showing a pattern of exponential growth in example 1?*
   a. The growth factor is 3.
   d. The growth factor is 1.5.

3. *The growth of a mold patch is recorded in the table below. The area covered by the mold is recorded every 24 hours. When the (time, area) pairs are graphed they seem to follow a pattern of exponential growth. Assuming that the pattern of growth is exponential, what is the* ***growth factor****?*

Time in days	0	1	2	3	4
Area in square cm	3.2	4	5.1	6.24	7.8

By dividing each area by the prior area value we have $\frac{4}{3.2}$ = 1.25, $\frac{5.1}{4}$ = 1.275, $\frac{6.24}{5.1}$ = 1.22, $\frac{7.8}{6.24}$ = 1.25. The ratio is not a constant. Since this data is collected by measuring in an experimental setting, we would not expect to find exactly the same ratio, even if an exponential pattern *is* the best fit to this situation. Therefore, averaging these growth factors would give approximately 1.25 as a growth factor.

Exponential Equations

Examining the growth pattern leads to a generalization that can be expressed as an equation: $y = ab^x$, where *b* is the growth factor, and *a* is the "starting" value, or the value of y when x = 0. The independent

4. *Give equations for the exponential relationships in example 1.*
   a. The growth factor is *3*. The **starting value** is *1*. The *x* tells how many times to multiply by 3. Therefore, the equation that fits this pattern is $y = 1(3)^x$.
   d. The growth factor is 1.5. The **initial value** is 100. The x tells how many times

variable, *x*, is an exponent; it tells how many times to multiply by the growth factor. (In the contexts in this unit x is a whole number. In general, x can be any real number.)

y-intercept or initial value: In the table the initial value is the value of y when x = 0. On the graph this appears as the y-intercept (see below.) Students can find the y-intercept or initial value from the equation or from the graph or from the table.

to multiply by 1.5. Therefore, the equation that fits this pattern is $y = 100(1.5)^x$.

5. *Find the equation that fits the exponential growth pattern shown in the table below.*

x	2	5	6	9
y	4500	15187.5	22781.25	*76886.71875*

In this example the table is incomplete; that is, it does not offer convenient values for y for every increment of 1 in x. This is not a problem as long as we are given that the underlying pattern is exponential growth. Comparing (5, 15187.5) to (6, 22781.25) we have a **growth factor** of $\frac{22781.25}{15187.5} = 1.5$. The other parameter we need is the **starting value**, that is the value of y when x = 0. To obtain this we can work backwards in the table, dividing by 1.5 at each stage. This produces

X	0	1	2	…
Y	2000	3000	4500	

Now we know the starting value and the growth factor. The equation that fits this exponential relationship is $y = 2000(1.5)^x$.

Graphs of Exponential Growth Relationships:

Have a characteristic shape, which is created by the **multiplicative** nature of the growth. In the graph of the *exponential* equation $y = ab^x$ the starting value or y –intercept is y = a when x = 0; at x = 1 we have y = ab; at x = 2 we have $y = ab^2$. In other words the graph of $y = ab^x$ will show a vertical change of $ab - a = a(b - 1)$ between x = 0 and x = 1; the vertical change will be $ab^2 - ab = ab(b - 1)$ between x = 1 and x = 2 etc. The vertical change is not constant. It increases by a multiplicative factor of b every time x increases by 1 unit. Compare this to

6. *Compare the graphs of $y = 2x + 1$ and $y = 2^x$. Specifically how does the vertical change in y values indicate that the first equation is linear and the second is exponential.*

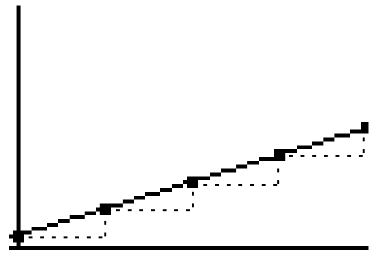
In the graphs of $y = 2^x$ (top graph) and $y = 2x + 1$ (bottom graph), the horizontal change is the same. Both graphs show a y intercept at (0,1).

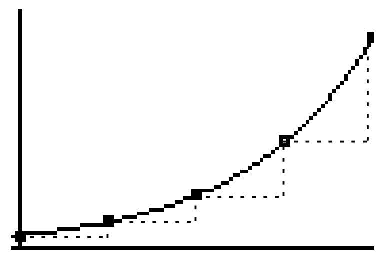
On the graph of $y = 2x + 1$, the vertical change is a constant; **2 is added** each time to the y-value, as x increases by 1 unit. This produces a straight line, slope 2, characteristic of a linear function. On the graph of $y = 2^x$ the vertical change increases by a **multiple of the growth** the graph of the *linear* equation y = mx + b. The value of y increases by adding *m* for each increment of 1 unit in x. The y-intercept is b. Thus starting at x = 0 we have y = b; at x = 1 we have y = b + m; for x = 2 we have y = b + 2m etc. In other words the graph of y = mx + b will show a vertical change of *m* for each increase of 1 unit in x.

Note: Exponential relationships can also be defined for negative and non-integer exponent values, though students do not work with these values for the independent variable in this unit. The related graphs are *continuous curves* (rather than graphs of plotted points) with shapes similar to those shown above.

The graph of any exponential growth pattern, $y = a(b)^x$ may show a slow increase at first but grows at an increasing rate because its growth is multiplicative. The graph curves upward from left to right.

increases by a **multiple of the growth factor** 2 as the graph rises; thus, first the vertical change is 2, then 2 x 2, then 2x 2 x 2 etc. This pattern of increasing change is characteristic of an exponential function.





Growth Rates

In this unit, **growth rate** is used the way we see that terminology in everyday settings, to compare a current value to a starting value by indicating the **percentage** by which the starting value has changed. This is different from *growth factor*. The two are related in a particular way.

In general, growth rate + 1 = growth factor. For example, growth rate of 5% is the same as a growth factor of 100% + 5% = 105%.

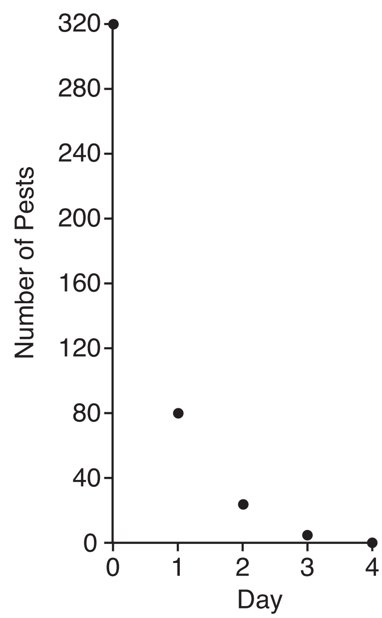
7. *If the cost of a cell phone increases from $80 to $120 we can say that the increase is $40. What is this as a **growth rate**?*

We can say that the percentage increase is $\frac{40}{80}$ = 50%. As a growth rate this is 0.5.

8. *The following table indicates the rise in a family's medical expenses over the past 3 years (numbers have been simplified). Is this an exponential relationship? If so what is the **growth factor** and what is **the growth rate?***

Time	0	1	2	3
Prices	$100	118	139.24	164.30

	$\frac{118}{100} = \frac{139.24}{118} = \frac{164.30}{139.24} = = 1.18$. Since there is a constant growth factor we may conclude that this data can be represented by an exponential relationship. The **growth factor** is 1.18 or 118%. Of this 118%, 100% represents the original price and 18% represents the percentage increase. Thus the **growth rate** is 18%. 9. *An investment offers an effective **growth rate** of 6%, on a starting amount of $1000. What are the values of the investment in the first 3 years? What is the **growth factor**? And what is the equation that fits this relationship between values and time?* The growth rate is given as 6%. This means that the value of the investment is 6% higher each year than it was the prior year. We can calculate the value of the investment each year by taking 6% of the prior year value and adding this on, or we can take 106% of the prior year's value. Thus, we can calculate 0.06(1000) + 1000 = 1060, or 1.06(1000) = 1060. <table><tr><td>Time</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>Value</td><td>1000</td><td>1060</td><td>1123.60</td><td>1191.02</td></tr></table> The **growth factor** is 1.06, and the starting value is 1000. So the equation is $V = 1000(1.06)^t$.
Exponential Decay Exponential models also describe patterns in which the value of a dependent variable decreases as time passes. In this case, the constant multiplicative factor is referred to as the ***decay factor***. Decay factors work just like growth factors, only they result in decreasing relationships because they are less than 1.	10. *An agricultural student is testing a new pesticide. The pesticide claims to reduce the number of crop eating insects by 80% at each application. The student estimates the number of pests and then sprays a test plot once a day. She does this for 5 days. Here are the results.* <table><tr><td>*day*</td><td>*0*</td><td>*1*</td><td>*2*</td><td>*3*</td><td>*4*</td></tr><tr><td>*Number of pests*</td><td>*320*</td><td>*80*</td><td>*24*</td><td>*6*</td><td>*0*</td></tr></table>*Do these results confirm the claim?* The graph of the pairs (day, pests) looks like

	the shape of the graph of an exponential relationship (see below). The point (4, 0) is an x-intercept and exponential relationships of the type $y = ab^x$ will not have an x –intercept. Since these are observational results we will concentrate on the general shape.  However, comparing succeeding “pest” values we have $\frac{80}{320}$ = 0.25, $\frac{24}{80}$ = 0.3, $\frac{6}{24}$ = 0.25. It appears that the **decay factor** is approximately 0.25. Thus the equation would be $p = 320(0.25)^d$. This indicates that every day the *number of pests left is 25%* of what was there the previous day, or, that the **decrease or decay rate** in the number of pests is 75%. The claim of 80% is too high. (Of course, new pests might be arriving every day, so that we could have an 80% decrease offset by a 5% increase.) Note: The 80% decrease rate is a typical way to indicate decrease situations. It relates to the decay factor the same way that growth rate relates to growth factor. *1 – decay rate = decay factor.*

It is helpful to think of the decay rate as the value *lost* and the decay factor as the value *left*. Thus an 80% decay rate means that 80% of the value is lost and 20% is left.

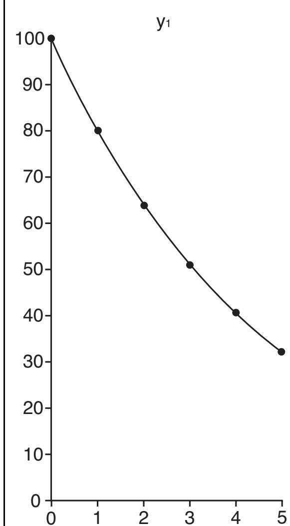
Graphs of Exponential Decay Relationships

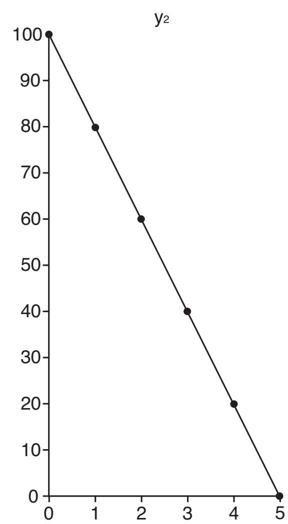
The basic pattern of exponential decay involves change from one point in time to the next by some constant factor. For decay, the change factor is between 0 and 1 and the graph curves downward from left to right, approaching the *x*-axis but never reaching it.

11. *The following is the graph of $y_1 = 100(0.8)^x$.*
    a. Compare to the graph of $y_2 = 100 - 20x$.
    b. Explain why there is no x intercept for y_1.

a.

X	0	1	2	3	4	5
Y_1	100	80	64	51.2	41	33
Y_2	100	80	60	40	20	0





The graph of y_2 shows a constant vertical change of 20 in y values for each change of 1 in x. y_2 is a linear function.

The graph of y_1 shows a *changing* vertical change in y. First the drop is 20, from x = 0 to x = 1. Then the drop is 16, or 0.8(20), from x = 1 to x = 2. Then the drop is 12.8, or (0.8)(16), from x = 2 to x = 3. In other words the vertical change is decreasing by a factor of 0.8 for each increase of 1 unit in x. This makes the curve steep to start with and less steep as x increases.

b. To continue the table for y_1 we have to multiply each y value by 0.8 to find the succeeding y value. Each y-value is smaller than the last, but can never be zero because

	it is always 80% of some positive quantity. (In practical situations we might say the amount left is zero because it is too small to be detected.)