Vocabulary: *Looking For Pythagoras*

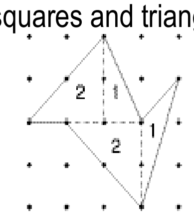
<table>
<tr><th>Concept</th><th>Example</th></tr>
<tr><td>

Finding areas of squares and other figures **by subdividing or enclosing:** These strategies for finding areas were developed in *Covering and Surrounding*. Students review finding areas by partitioning the figure into shapes whose areas they can easily find, or by surrounding the figure with a rectangle or square and subtracting the “extra” areas.

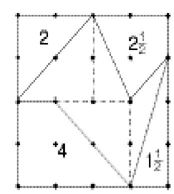
Note: the reason for this review is to have students develop a strategy for finding the length of the side of a square, if the area CAN be calculated but the length of a side can NOT be calculated with existing knowledge.

</td><td>

1. In the figure on the left below, the shape is divided into 4 triangles, the areas of which can be found by using the given base and height. (See *Covering and Surrounding* for area of triangle.) In the figure on the right the same shape is surrounded by a square whose edges are each 4 units, so the square has area 16 square units. We must now subtract the “extra” areas, some of which are triangles, and some of which are shapes that can be divided into squares and triangles.

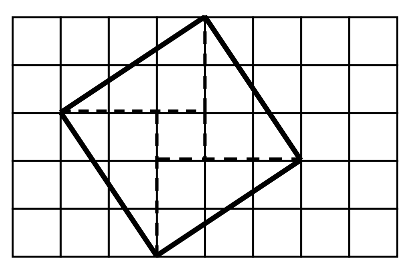


Subdivide to find the area: $2 + 2 + 1 + 1 = 6$



Enclose in a square to find the area: $16 - (4 + 2\frac{1}{2} + 2 + 1\frac{1}{2}) = 6$

2. The bolded square below does not “line up” against the grid lines and so it is hard to calculate, or see its area. However, if we draw broken lines, as shown, we can figure out that the area is actually made of 4 right triangles, with area 3 square units, and 1 small square with area 1 square unit. The total area of the square is 13 square units.

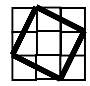


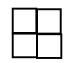
Note: this area could also have been calculated by surrounding the bolded square with a rectangle.

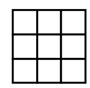
</td></tr>
</table>

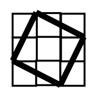
<table>
<tr>
<td>Square Root: can be thought of as the length of the side of a square whose area is known. Thus, a square with area 9 square units has a side of length square root of 9, or 3 units. We write “square root of 9” as $\sqrt{9}$. Or, it can be thought of as a number which when multiplied by itself gives a target number. Thus, to evaluate $\sqrt{20}$ we need to find a number which when multiplied by itself yields 20.

Benchmarks: are useful when trying to evaluate square roots. Thus, $\sqrt{20}$ must be greater than 4, because 4 is $\sqrt{16}$, but less than 4, because $\sqrt{25}$ is 5.

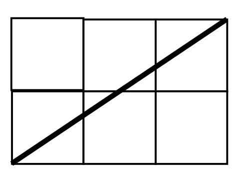
Since many square roots are irrational (see below) students can not calculate them exactly. They will either rely on benchmarks, or on a calculator.</td>
<td>3.

Area of bolded square = Area of outer square – 4 triangles
= 9 square units – 4(1) square units
= 5 square units
So length of side of square = $\sqrt{5}$ units.

4.
2 x 2 = 4 

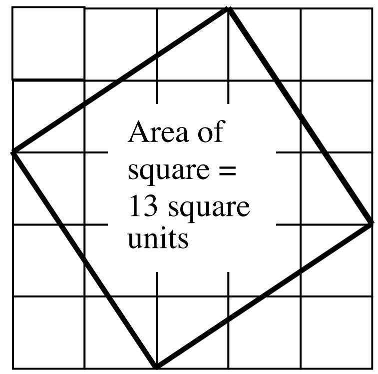
3 x 3 = 9 

The area of the bolded square below is 5 square units (see example 3). So each side has length $\sqrt{5}$ units.
? x? = 5 

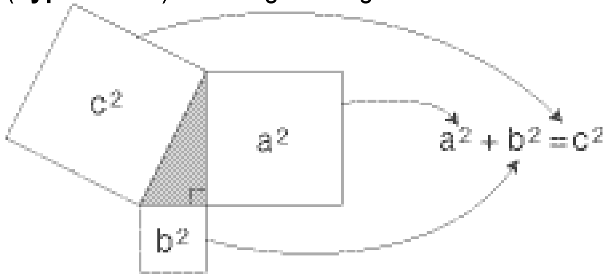
Since 4 < 5 < 9, taking square roots, $2 < \sqrt{5} < 3$.

Comparing the areas of the square figures above, we would guess that$\sqrt{5}$ is closer to 2 that to 3. Students might guess and check, using calculators or multiplying by hand. 2.1 x 2.1 = 4.41, 2.2 x 2.2 = 4.84, 2.3 x 2.3 = 5.29. Apparently $\sqrt{5}$ is between 2.2 and 2.3. (Using a calculator we find that $\sqrt{5}$ is approximately 2.236. $\sqrt{5}$ is irrational, so there is no exact terminating decimal equal to $\sqrt{5}$.)</td>
</tr>
<tr>
<td>The relationship between square area and side length: Since the side length of a square is the square root of the area of a square, students can find areas, using partitioning or surrounding strategies, and then use this to find the side length.</td>
<td>5.


The length of the above line segment could be measured with a ruler; since all measurement is an approximation this would give us some idea of the length. To calculate an exact length,</td>
</tr>
</table>

	assuming the line segment connects two vertices on the grid, we could construct a square on this line segment, find the area of the square (using partitioning or surrounding as a strategy) and then calculate the length of a side.  So length of side of square = exactly √13, or approximately 3.6 units.

Pythagoras Theorem: says that the sum of the square areas on the two shorter sides of a right triangle is the same as the area on the longest side (**hypotenuse**) of the right triangle**.**

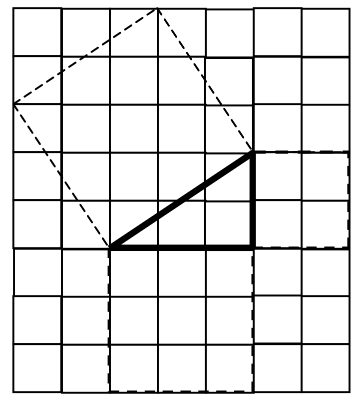


Students discover this pattern when they build squares on the sides of a right triangle, and then find the areas as in example 5 above. They also investigate a proof that this pattern works for all right triangles, and only for right triangles.

The **converse of the Pythagoras Theorem** states that if the sum of the areas of the squares on the two shorter sides of a triangle is the same as the square area on the longest side of the triangle, then the triangle must be a right triangle. Note that the original theorem starts with a given right triangle and proves the relationship between the square areas. The converse starts with the given relationship between the square areas and proves the triangle must be right angled.

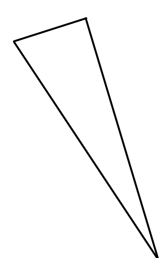
Pythagorean Triples: are sets of three whole numbers that fit the Pythagorean relationship, and therefore form right triangles. For example, 3 – 4 – 5 is a Pythagorean Triple, because $3^2 + 4^2 = 5^2$. Therefore we can form a right triangle with these lengths or with any scaled up copy (see *Comparing and Scaling*) of these lengths. The triple 3 – 4 – 5 is really a ratio 3:4:5, since any multiple of 3 – 4 – 5 will also be a Pythagorean Triple. In fact all right triangles formed by the triple 3 – 4 – 5 will be similar. There is an infinite number of these Pythagorean Triples. 5:12:13 is another example.

6.

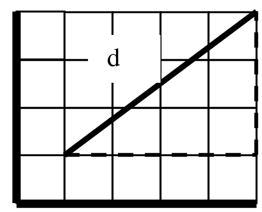


The original triangle has sides 2, 3 and a hypotenuse of unknown length. The areas of the squares on the sides are 4 and 9 square units. The area of the square on the hypotenuse can be calculated as in example 5, 13 square units. For this example we can see that the sum of the areas of the squares on the two sides of a right triangle (4 + 9 square units) is the same as the area of the square on the hypotenuse (13 square units). *Note: This is only one example, and should not be regarded as a proof. Students do a very visual proof using an arrangement of triangles and squares to show that the sum of the square areas on the short sides of* ***any*** *right triangle is the same as the area of the square on the hypotenuse..*

7. Is the following triangle right angled? Lengths of sides are a = 2.5, b = 6 and c = 6.5 units.

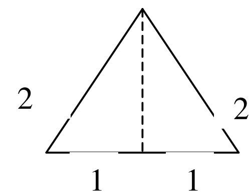


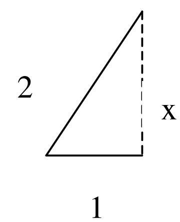
We could measure all the angles in the triangle, but this would be an approximation of angle sizes. We can *calculate* squares of side lengths

	as follows: $a^2 = 2.5^2 = 6.25$. $b^2 = 6^2 = 36$. $c^2 = 6.5^2 = 42.25$. Since $a^2 + b^2 = c^2$ we can deduce that this triangle is right angled, with the right angle opposite the longest side, c. 8. Find the distance between two points on a coordinate grid.  The above sketch shows a line segment joining two points on a coordinate grid. The points are (1, 1) and (5, 4). To find the distance between these two points we can create a right triangle, and apply the Pythagorean Theorem. $d^2 = 3^2 + 4^2 = 25$. Therefore, d = 5.

Special right triangles: A triangle with angles 30, 60 and 90 degrees will have side lengths that satisfy the Pythagorean relationship; a triangle with angles 45, 45 and 90 degrees will have side lengths that satisfy the Pythagorean relationship. The side lengths of any 30-60-90 triangle are in the ratio 1: √3: 2; the side lengths of any 45-45-90 triangle are in the ratio 1: 1: √2.

9. To see why a 30-60-90 triangle has sides in a particular ratio we first examine a 60-60-60 triangle with each side length 2 units. Notice that the altitude (at right angles to the base) bisects the base into two lengths, each 1 unit, creating two 30-60-90 triangles.

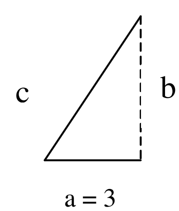




The sides of this 30-60-90 triangle satisfy the Pythagorean relationship and so,
$1^2 + x^2 = 2^2$, so
$x^2 = 3$, so
$x = \sqrt{3}$.

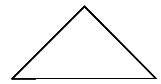
The side lengths are 1, √3, 2 units

10. Triangle ABC, sketched below, has angles 30, 60, 90 degrees, and the shortest side length is 3 units. What are the other side lengths?



This triangle is a scaled up similar copy of the triangle in example 9. (See *Stretching and Shrinking*.) The scale factor is 3. So, the lengths are 3(1), 3(√3), and 3(2) units, or 3, 3√3, 6 units.

11. Triangle PQR is a 45-45-90 triangle. The hypotenuse is 5 units long. How long are the other sides?

	 q = 5 The ratio of side lengths for a 45-45-90 triangle is 1:1:√2. In this triangle, which is a similar copy of every other 45-45-90 triangle, the ratio is p: r: 5, where p and r are the equal sides. We can think of using a scale factor of 5/(√2) to scale up a triangle with sides that measure 1, 1, √2 units. This will create a triangle with sides that measure 5/(√2), 5/(√2), 5 units, or approximately 3.5, 3.5, 5 (Note: students could also find the sides by using some algebraic reasoning. $P^2 + r^2 = q^2$, so $p^2 + p^2 = 25$, so $2p^2 = 25$, so $p^2 = 12.5$, so p = √12.5. or approximately 3.5 units.)

Rational numbers: are any numbers that can be written in the form a/b where a and b are integers, but b can not be zero. Students can think of these as anything that can be written as a positive or negative fraction.
Note: every rational number can be written as a decimal, either terminating or repeating. (See Vocabulary, *Bits and Pieces III*.)

Irrational numbers: are numbers that can NOT be written in the form a/b where a and b are integers. Non-repeating, non-terminating decimals, and square roots that do not work out exactly and ▯ are examples of irrational numbers.
Note: since numbers like $\sqrt{2}$ and $\sqrt{5}$ are irrational any decimal approximation will be inexact, no matter how many decimal places we use. $\sqrt{2} = 1.4142\ldots$ and $\sqrt{5} = 2.2360\ldots$ The decimal approximations never terminate and never repeat. If they did terminate or repeat then these decimals could be written as rational numbers; but $\sqrt{2}$ and $\sqrt{5}$ are irrational numbers. Using the format $\sqrt{2}$ is exact, whereas 1.4142 is a very accurate, but inexact, approximation.

Real #'s: are all the numbers which are either rational or irrational.

Note: *every* number that students know about at this stage is a real number. In High School they will meet other kinds of numbers, such as complex numbers.

12. Which of these numbers are rational numbers:
2, 2.4, 0.1111…, -9, 2 1/3, 17/5, -2/7?

ALL of these numbers are rational. They CAN all be written as a/b.
2 = 2/1
2.4 = 24/10 (every terminating decimal can be written as a fraction with a power of 10 for a denominator)
0.1111… = 1/9 (see below)
-9 = -9/1
2 1/3 = 7/3
17/5 is already in the "a/b" format.
-2/7 is already in the "a/b" format.
From the above examples we can conclude that any integer, any positive or negative fraction, or mixed number, and any terminating decimal can be written as a rational number.

13. a. Write 1/9 as a decimal.
Every fraction can be thought of as a division. So 1/9 can be thought of as 1 ÷ 9. We can set this up as a division, 1.0000 ÷ 9, and get the decimal answer, 0.1111… (See *Bits and Pieces III* for decimal division.)

b. Write 0.121212…as a rational number.
We can think of this as an algebra problem.
X = 0.121212…
So, 100x = 12.121212…
So, 100x – x = 12.121212…. – 0.121212….
= 12.
(Notice there is no repeating part now.)
So, 99x = 12. So, x = 12/99.
This strategy could have been used for any repeating decimal. *Any repeating decimal can be written as a rational number.*

14. Give an example of a non-terminating and non-repeating decimal.
0.3 is a terminating decimal. 0.333…is a repeating decimal. But 0.32332333233332… has a pattern which neither terminates nor repeats. Thus 0.32332333233332…is an irrational number.