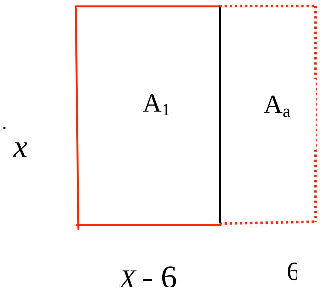
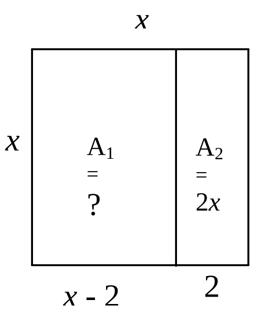
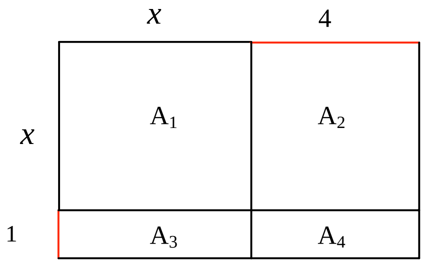
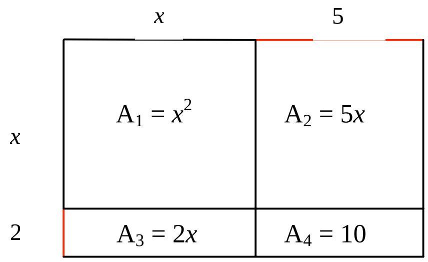
Selected ACE: *Say It With Symbols*

Investigation 1: 19, 22, 27, 31.
Investigation 2: 6, 7, 8, 23, 29.
Investigation 3: 21, 25.
Investigation 4: 8 – 10, 18.
Investigation 5: 7, 12.

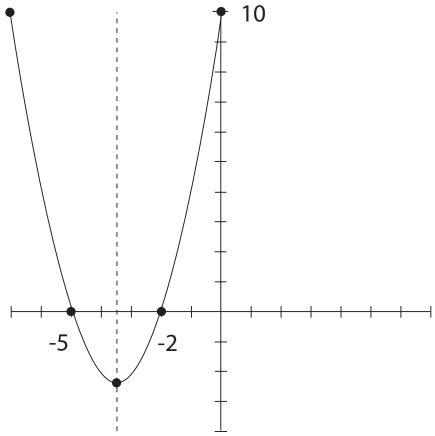
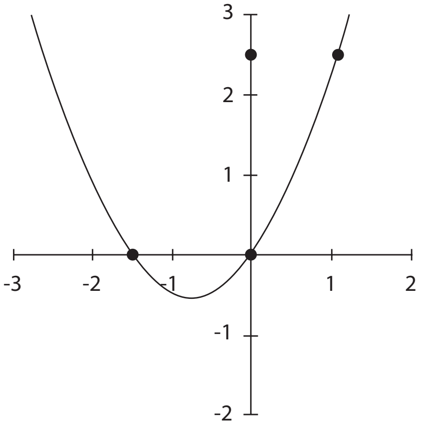
ACE Problem	Possible solution
Investigation 1	
19. Draw a divided rectangle for the expression. Label the lengths and the area. Write an equivalent expression in expanded form. $x(x - 6)$	The *area model* (see *Frogs and Fleas*) is familiar to students and makes a useful visual representation of the product of 2 factors, one of which is the length and the other is the width. 19. Suppose that x is the width of this rectangular area, then the length has to be $x - 6$, and the area is $x(x - 6)$.  The rectangle on the left has the required length and width. It has area $A_1 = x(x - 6)$. The rectangle on the right has length 6 and width x; so $A_2 = 6x$. The two rectangles together make a square (outlined in red) with each side length x, so the area of the red square = x^2. Now A_1 = large square – A_2. So $x(x - 6) = x^2 - 6x$. As students become more familiar with the Distributive Property they can rewrite any expression from factored form to expanded form. The area model serves as an initial explanation and bridge to the manipulation of the symbols.
22. Draw a divided rectangle for the expression. Label the lengths and the area. Write an equivalent expression in expanded form. $x^2 - 2x$	22. If we try to make sense of the symbolic expression then we see that we have a “square” minus a “2 by x rectangle”. As an area model this looks like:

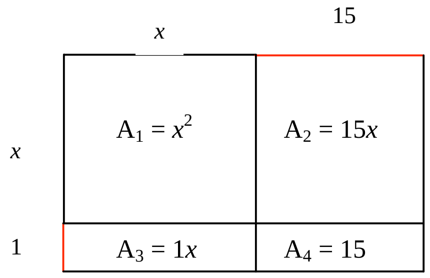
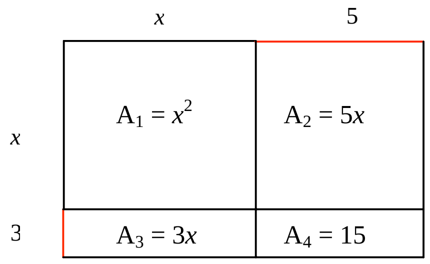
	 The area A_1 is the area we want, the large square minus the rectangle with area A_2. A_1 = large square – $A_2 = x^2 - 2x$. A_1 can also be represented by length times width = $x(x - 2)$. So $x^2 - 2x = x(x - 2)$.
27. Draw and label a rectangle whose area is represented by the expression. Write an equivalent expression. $(x + 1)(x + 4)$.	27. The two factors are $x + 1$ and $x + 4$. One factor represents the length of the rectangle, and one factor represents the width.  The total area is $(x + 1)(x + 4)$. This is equivalent to the sum of several smaller areas: $A_1 + A_2 + A_3 + A_4 = x^2 + 4x + 1x + (1)(4) = x^2 + 5x + 4$. So, $(x + 1)(x + 4) = x^2 + 5x + 4$.
31. Draw and label a rectangle whose area is represented by the expression. Write an equivalent expression. $x^2 + 7x + 10$.	31. The difficult part of this problem is thinking how to split up the “10” into factors which are parts of the *dimensions* of the rectangle, while having the “$7x$” be parts of the *area* of the rectangle. Students might proceed by guessing and checking, making several drawings of rectangles. It makes sense to check *factors* of 10, like 1 times 10 and 5 times 2. The correct pair of factors is 5 times 2. These appear on the area model below as parts of the length and width.

	 With this arrangement we see that the sum of the areas is the required $x^2 + 7x + 10$. And we can also write the same area as length times width $= (x + 5)(x + 2)$. So, $x^2 + 7x + 10 = (x + 5)(x + 2)$.
Investigation 2	
6. Recall from Problem 2.2 that: $\boldsymbol{P} = 2.50V - 500$ and $\boldsymbol{V} = 600 - 500\boldsymbol{R}$. Suppose the probability of rain is 50%. What profit can the concession stand expect to make?	6. In this example we cannot find the profit P directly from the probability of rain R, because we have, as yet, no equation linking R and P. So we have to solve this problem in 2 stages. If $R = 0.5$, then V (number of visitors) $= 600 - 500(0.5) = 350$. If $V = 350$, then P (profit) $= 2.50(350) - 500 = 375$. We expect a profit of $375. [Alternatively: we could make an equation connecting P and R by combining the two given equations. Since we know that $V = 600 - 500R$, we can substitute this expression for V into the other equation: $P = 2.50(600 - 500\boldsymbol{R}) - 500$. This equation could be rewritten in an equivalent form, but it is also usable as it stands. The good thing about this equation from our point of view is that it has only 2 variables, P and R. Substituting $R = 0.5$ into the equation we get: $P = 2.50(600 - 500(0.5)) - 500$. This will give the same answer as above, but in one calculation.]
7. (See equations in #6) What is the probability of rain if the profit expected is $100?	7. This asks students to reverse the reasoning from #6. Because we do not have a single equation relating R and P we have to solve this in 2 stages. This time we can substitute 100 for P, and solve

	$100 = 2.50V - 500$. This gives the number of visitors V needed to make a profit of \$100. (Not done here.) We still have to substitute that answer for V into the second equation and solve for R to find out what probability of rain is associated with the required number of visitors. [Alternatively, use the equation $P = 2.50(600 - 500R) - 500$, and substitute $P = 100$, then solve for R.]
8. See #6 for equations. The manager estimates the daily employee bonus fund B (in dollars) from the number of visitors V using the equation $B = 100 + 0.50V$. a. Suppose the probability of rain is 30%. What is the daily employee bonus fund? b. Write an equation that relates the employee Bonus B to the probability of rain. c. Suppose the probability of rain is 50%. Use your equation to calculate the employee-bonus fund. d. Suppose the daily employee bonus fund is \$375. What is the probability of rain?	8. a. We have to use 2 equations here: $B = 100 + 0.5V$ and $V = 600 - 500R$. If we do this in 2 stages we have $V = 600 - 500(0.3) = 450$. This says that when the probability of rain is 30% we can expect 450 visitors. Now use this in $B = 100 + 0.5V$ to find out what the employee bonus will be. (Not done here.) b. We can combine the 2 equations used in part a, by substituting the expression for V from the second equation into the first equation. (See alternative explained in #6 above.) You should get an equation that has only 2 variables, B and R, starting B = 100 + 0.50(……). c. This can be answered by substituting 0.5 for R in the equation you found in part c. (Not done here.) d. Assuming that you found the correct equation in part b, you should now be solving: $375 = 100 + 0.5(600 - 500R)$ for R. You will probably want to use the Distributive Property to rewrite this equation as $375 = 100 + 300 - 250R$ before you start applying the *Properties of Equality* (see #23 below).
23. Use *Properties of Equality* to solve: $9 - 4x = \frac{3+x}{2}$	23. *Properties of Equality* allow us to rewrite an equation by adding or subtracting or multiplying or dividing on both sides of the equation. As long as we do the same to both sides of the equation the equals sign is still true. The goal is to use these properties to produce simpler, but equivalent, equations. $9 - 4x = \frac{(3+x)}{2}$ We might begin this by multiplying both sides by 2. $2(9 - 4x) = \frac{2(3+x)}{2}$

	The point of doing this is to get rid of the fraction on the right side. $18 – 8x = 3 + x$. Now we might add $8x$ to each side, producing: $18 = 3 + 9x$. Students have practiced solving this type of equation many times, so they should have no trouble getting the solution: $X = \frac{15}{9}$. Alternatively, we might handle the fraction on the right side more directly. Applying the Distributive Property we have: $9 – 4x = \frac{3}{2} + \frac{x}{2}$, or $9 – 4x = 1.5 + 0.5x$. Again students should have no trouble completing the solution. They first learned how to solve equations like this *in Moving Straight Ahead*, and reinforced this many times since then, particularly in *Thinking With Mathematical Models.*
29. For the equation $y = (x + 2)(x + 5)$, find each of the following. Explain how you found each. a. y-intercept b. x-intercept(s) c. maximum or minimum point. **d.** line of symmetry.	29. This equation is not a linear equation; it is a quadratic equation in factored form. Students studied quadratic equations in *Frogs and Fleas*. They know that the graphs of quadratic equations are parabolas. In this unit they have used an area model and the Distributive Property to rewrite a quadratic expression in factored form into an equivalent quadratic expression in expanded form. a. The y-intercept is the point where $x = 0$. Substituting in this equation we have $y = (0 + 2)(0 + 5) = 10$. (Alternatively, students might be working with the equivalent equation $y = x^2 + 7x + 10$.) b. To find the x-intercepts students might use their graphing calculators to look at the graph or table for points where $y = 0$. Or they might solve $0 = (x + 2)(x + 5)$. The only way that a product of factors can result in zero is if one or both of the factors is zero. Thus there are two possible values of x that will give zero for y. • If $x = -2$ we have $y = (-2 + 2)(-2 + 5) = 0$, and • if $x = -5$, we have $y = (-5+2) (-5 + 5) = 0$. Therefore there are 2 values for x that give $y = 0$. The x-intercepts are $x = -2$ and $x = -5$. C and d. In *Frogs and Fleas* students learned to use the symmetry of the graph to find the line of symmetry and the vertex. The line of symmetry will cross the x-axis halfway between the x-intercepts, that is halfway between $x = -2$

	and x = -5, at x = -3.5.  Since the vertex is ON this line of symmetry and ON the parabola we must look for the y-value to complete the point (-3.5, ?). Substituting x = -3.5 into the equation for the parabola will give this point. (Not done here.)
Investigation 3	
21. Solve for x. Check your solution using calculator tables or graphs. $x^2 + 1.5x = 0$.	21. $x^2 + 1.5x = 0$ is a quadratic equation. When students see this type of equation they will probably visualize a parabola, representing a relationship, $y = x^2 + 1.5x$. The solutions of the given equation can be found from this graph.  We want the points where y = 0; that is (x, 0). There are two points or x-intercepts: (0, 0) and (-1.5, 0).

	So $x = 0$ and $x = -1.5$ are solutions. (Students might also use a table to find these solutions.) [Alternatively, students might rewrite this equation in factored form: $x(x + 1.5) = 0$. We can deduce from this form that the only way that the product of the two factors can be zero is if either one or both are zero. Therefore, we can deduce that either $x = 0$, or $x + 1.5 = 0$. This gives us the same two solutions: x could be 0 or -1.5.]
25. Use an area model to factor a. $x^2 + 8x + 15$.	25. Students have to find the dimensions of a rectangle whose area is $x^2 + 8x + 15$. The first clue is the constant term: 15. We need factors of this term. These could be 15 times 1 or 5 times 3. Using 15 and 1 as factors of 15 results in the first arrangement below. Notice that the area of the rectangle does NOT sum to $x^2 + 8x + 15$.   The second arrangement DOES sum to the desired $x^2 + 8x + 15$. Therefore, $x^2 + 8x + 15$ can be rewritten as $(x + 5)(x + 3)$.
Investigation 4	
8 – 10. See student text for graphs.	8. Students should recognize the characteristic graph *shape* for each type of function. They know that *linear functions (see*

<table>
<tr>
<td>8. Which graphs represent linear, quadratic, or exponential functions?

9.
Make a table of y values for x = 1, 2, 3, 4, 5, 6 for each function.

10.
Write an equation for each function. Describe your strategy.</td>
<td>*Moving Straight Ahead)* are represented by *straight line graphs* (graph 1), and that *quadratic functions* (see *Frogs and Fleas*)are represented by *parabolas* (graph 3). They also know that *exponential growth functions* (see *Growing, Growing*) are represented by graphs that show y-values increasing multiplicatively (each y-increase is related to the last y-increase by the same factor). The remaining graph is actually an *inverse proportion relationship*, where increases in x values cause decreases in y-values. Students studied this type of relationship in *Thinking With Mathematical Models*.

9. (Only graph 3 is addressed here.)
The table of values for the *quadratic relationship* should show that points are symmetrically placed around the axis of the parabola. Students should also know that the differences in the y-values make a pattern, and that the differences of the differences (*Second Differences*, see *Frogs and Fleas*) will be a constant. This makes it possible to figure points that are not shown on the graph. Thus for graph 3:
<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>y</td><td>0</td><td>2</td><td>2</td><td>0</td><td>-4</td><td>-10</td><td>-18</td></tr>
<tr><td></td><td>y-inc of 2</td><td>Y –inc of 0</td><td>y-dec of 2</td><td>y-dec of 4</td><td>y-dec of 6</td><td>y-dec of 8</td><td></td></tr>
</table>
Continuing the pattern of change in the y-values we see that between $x = 5$ and $x = 6$ the y-values must decrease by 8, which means that when $x = 6$ the corresponding y-value must be -10 – 8 = -18. (In this example the second differences are all -2.)

10.
Graph 1 represents a *linear relationship*. The *slope* of this graph is $\frac{3}{1}$, so the *constant rate of change* for y, compared to x, is 3. We can also see that the *y-intercept* is -2. Putting this information into the general slope-intercept format of a linear equation, $y = mx + b$, gives us the equation $y = 3x - 2$. We can check that all the points on the graph do indeed fit this equation.</td>
</tr>
<tr>
<td>**18.**
For parts a – c, use the set of equations below.
1. $y = x^2 + 8x$.
2. $y = 2x$.
3. $y = 4^{x-1}$
4. $y = 2(x - 3) + 6$.
5. $y = x(x + 8)$</td>
<td>18.
a. *Linear equations* must all fit the format, $y = mx + b$. However, they may need to be rewritten to fit this format. Generally we are looking for equations in which we are sure that there is no power of x higher than 1. Equation #2 obviously fits this format. But so also does equation #4. If you use the Distributive property to rewrite equation #4 you find it is:</td>
</tr>
</table>

6. $y = 0.25(4^x)$ 7. $y = 0.25^x$ 8. $y = 17 + x(x + 3)$ 9. $y = (x + 1)(x + 17)$ a. Which equations represent linear, quadratic or exponential functions. b. Find any equations that represent the same function. c. Without graphing the equation, describe the shape of the graph of each equation in part b.	$y = 2x – 6 + 6$, or simply $y = 2x$, equivalent to equation #2. *Quadratic equations* must all fit the *expanded format* $y = ax^2 + bx + c$, or the *factored format* $y = (ax + b)(cx + d)$. The first format is recognizable for the "x^2" term, and the second format is recognizable for its pair of *linear* factors. Of course, an equation is still considered to be quadratic if it is not in either of these forms, but CAN BE rewritten in one of these forms. Thus, equation #1 is a quadratic in expanded form and equations #5 and #9 are equations in factored format. In addition there is *one other* quadratic equation, but it has to be rewritten to spot the "x^2" term. *Exponential equations* all fit the format $y = a(b)^x$. There are three exponential equations here. b. There are matches for equations 1, 2, and 3. (Students should be able to apply the Distributive Property to make the match for equations 1 and 2.) Equation 1 is a *quadratic equation* and one *other quadratic equation* is identical to it. Equation 2 is a *linear* equation and one other *linear* equation is identical to it. Equation 3 is an *exponential equation* and one other *exponential equation* is identical to it. (Students may need to make a table to see which two of the three exponential equations are identical.) c. See #8 above for information about graph shapes.
Investigation 5.	
7. Look at the product of three consecutive whole numbers. For example: 1 x 2 x 3 = 6. 2 x 3 x 4 = 24, 3 x 4 x 5 = 60. a. What pattern do you observe? b. Make a conjecture about the product of three consecutive whole numbers. Explain why your conjecture is correct.	7. a. It looks like the product is always a multiple of 6. Checking a few other example we see that : 4 x 5 x 6 = 120, and 10 x 11 x 12 = 1320. The pattern *seems* to always work. BUT, checking examples is no guarantee or proof that the pattern holds for ALL products of three consecutive numbers. b. We can conjecture that the product of three consecutive numbers is always $6n$ for some integer n. ONE way that students might begin to think about this is to name the three consecutive numbers in some general way. We might observe that if the first is an even number then it could be written as $2x$, and the others would then be $2x + 1$

and $2x + 2$. The product $2x(2x + 1)(2x + 2)$ is obviously even, since it starts with a factor of 2, but it is not clear from this format that there would also be a factor of 6. If we think that maybe the first is an odd number and the second is even then we have
$(2x - 1)(2x)(2x + 1)$. Again this is obviously divisible by 2, but not so obviously divisible by 6 also.

To be sure that the product is divisible by 6 we need it to be divisible by 2 and by 3 (See *Prime Time*). As above we can see that the product must be divisible by 2. To prove that it is also divisible by 3 we need to be sure that in any arrangement of 3 consecutive numbers one of them must have a factor of 3. And indeed that is true:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12……

So, in our general product $2x(2x + 1)(2x + 2)$ or
$(2x - 1)(2x)(2x + 1)$ one of these factors MUST also be divisible by 3.

12.
Study the sequence of cube buildings below.

- What pattern do you notice?
- Use the pattern to construct the next building.
- Think about the steps as you construct your building. The labels below show you one way to think about your pattern.

See text for diagrams.

a. Describe a pattern you see in the cube buildings.
b. Use your pattern to write an expression for the number of cubes in the nth building.
c. Use your expression to find the number of cubes in the fifth building.
d. Use the Distributive and Commutative Properties to write an expression equivalent to the one in part

12.
a. The number of cubes is the sequence: 1, 6, 11…. It looks like the number of cubes increases by 5 each time.

b.

Building, n	1	2	3	4
# cubes, c	1 cube	1 cube + 5 arms of 1 cube	1 cube + 5 arms of 2 cubes	1 cube + 5 arms of 3 cubes

$c = 1 + 5$ arms of $(n – 1)$ cubes.
$c = 1 + 5(n – 1)$

c. If $n = 5$ then $c = 1 + 5(5 – 1) = 21$.

d. $c = 1 +5(n – 1)$
$c = 1 + 5n – 5$
$c = 5n – 4$.
This is a linear equation with a constant rate of change of 5 cubes. If we look at the drawings we should be able to see where 5 cubes are added each time as the *next* building is created from the last building.

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<td>b. Does this expression suggest another pattern in the cube buildings?
e. Look for a different pattern in the buildings. Describe the pattern and use it to write a different expression for the number of cubes in the nth building.</td>
<td>e. Students might see this as a tower surrounded by 4 wings. The tower grows 1 higher each time, and is always the same height as the building number; the wings also grow by 1 cube each time. With this arrangement in mind we can see
<table>
<tr><td>Building n</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td># cubes c</td><td>1 tower of 1 + No wings</td><td>1 tower of 2 + 4 wings of 1</td><td>1 tower of 3 + 4 wings of 2</td><td>1 tower of 4 + 4 wings of 3.</td></tr>
</table>
This suggests that the nth building has:
C = 1 tower of n + 4 wings of ($n - 1$).</td>
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