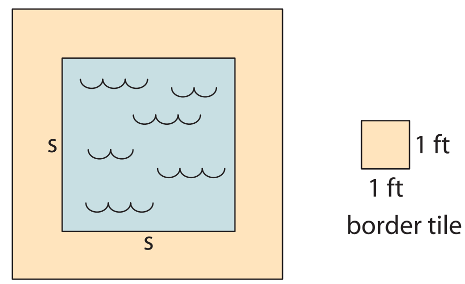
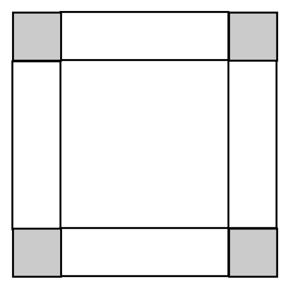
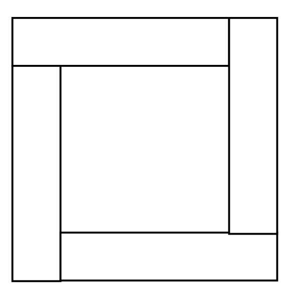
Vocabulary: *Say it with Symbols*

Concept	Example
Equivalent Expressions: are algebraic expressions that have the same value no matter what value(s) is substituted for the variable(s). Students have various ways to check for equivalence: using graphs and tables, or using Properties of Real numbers. See below. Note: Students often confuse “expression” and “equation.” “3x” is an *expression*. It can be *evaluated* for various values of x. However, “y = 3x” or “12 = 3x” are *equations*. These two equations have *solutions*. In the case of y = 3x there are an infinite number of solutions: (1, 3), (2, 6) etc. In the case of 12 = 3x there is only one solution, x = 4. Likewise: $2x^2 + 10x$ is an *expression* which can be *evaluated*: while $2x^2 + 10x = 12$ is an *equation* which can be *solved*.	1. *Write 2 expressions for the number of 1-foot-square tiles, N, needed to make a border around a square pool with sides of length s feet.*  The expression for N depends on how a student visualizes the area of the tiled border. A student might say that the border is made of 4 strips, each s tiles long, and 4 corners, as below.  In this case the student will say that the number of tiles needed is N = 4s + 4. Or the student might see this as 4 strips, each (s + 1) tiles long, as below.  In this case the student will say that N = 4(s+ 1) tiles are needed. 2. *Determine that the two expressions for N in*

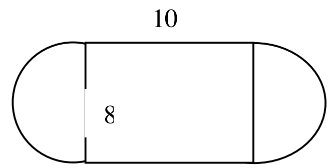
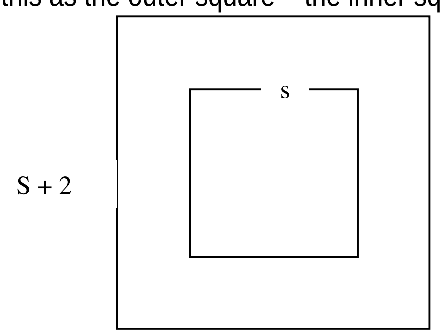
example 1 are equivalent.

Students already know the meaning of "equivalent" in terms of functions. So they could compare the expressions "4s + 4" and "4(s + 1)" by making a table, for various values of s.

S	1	2	3	4
4s + 4	8	12	16	20
4(s + 1)	8	12	16	20

It appears that these two expressions always have the same value, for any given value of s.

Note: Students could also use a graph to make this comparison. However, tables and graphs are only snapshots of SOME of the values that an expression might take. To be sure that these expressions are equivalent for ALL values of the variable we need a better strategy. See below.

Interpret an Expression: Students who develop familiarity with symbolic expressions can recognize what a particular expression might represent. Note: "3x" is a **monomial** because it has 1 term. "3x + 2y" is a **binomial** because it has 2 terms added. If an expression has 3 terms it is called a **trinomial.** An expression with more than 3 terms is called a **polynomial.** (You can use "polynomial" to refer to an expression with ANY number of terms.)	3. *Sketch a pool whose area is given by* *$16\pi + 80$ square feet.* There are several possible answers, but students should be able to spot a symbolic expression related to a circle within the given **binomial** expression. The **term**"16**π**" is the area of a circle with radius 4 units. The other **term**, "80", could be a rectangular area. One possible pool would be  4. *If a student gives the number of tiles needed for the pool problem as $N = (s + 2)^2 – s^2$, how did the student visualize the problem?* There are 2 squared expressions within this polynomial: $(s + 2)^2$ and s^2. The presence of the "square" implies a square area. "s^2" is the area of a square with sides s; "$(s + 2)^2$" is the area of a square with side (s + 2). On the sketch of the original pool the student could have visualized this as the outer square – the inner square. 
Properties of Real Numbers **The Commutative Property: of addition** states that the order of addition of real numbers does not matter. a + b = b + a for all real values of a and b. Note: Multiplication of real numbers also has this property. Subtraction and division of real numbers do not have this property. For	5. *Show that 4s + 4, 4(s + 1) and 2(s + 2) + 2s are* ***equivalent*** *by using properties of real numbers.* 4(s + 1) = 4s + 4 (using the **Distributive property** to multiply each of the terms of "s + 1" by 4.) 2(s + 2) + 2s = 2s + 4 + 2s (using the **Distributive Property** to multiply

example, 8 – 3 is not the same as 3 – 8.

The Associative Property: of addition of real numbers states that when adding 3 (or more) real numbers you may group them in pairs and add, using any groupings.
a + b + c = (a + b) + c = a + (b + c).
Note: Multiplication of real numbers also has this property, but subtraction and division do not. For example,
12 ÷ 4 ÷ 2 = (12 ÷ 4) ÷ 2 = 3 ÷ 2 = 1.5.
12 ÷ 4 ÷ 2 ≠ 12 ÷ (4 ÷ 2) = 12 ÷ 2 = 6.
There is a particular order of operations that we use when we have several operations to do to evaluate an expression. In this case we do the divisions in order from the left. As we have seen, if there are only additions of multiplications in the expression then we can change the order, or group them in any way. See *Accentuate the Negative* for more on **Order of Operations.**

Distributive Property:

- If an expression is written as a factor multiplied by a sum of two are more terms, the distributive property can be applied to ***multiply or expand*** the factor by each term in the sum.
- If an expression is written as a sum of terms and the terms have a common factor, the distributive property can be applied to rewrite the expression as the common factor multiplied by a sum of two or more terms. This process is called ***factoring***.

multiply

$$r(s + t) = rs + rt$$

factor

Note: The distributive property was first introduced in *Accentuate the Negative* and extended in *Frogs, Fleas, and Painted Cubes*, to include two binomials, (*a* + *b*)(*c* + *d*).

each of the terms of "s + 2" by 2)
= 2s + 2s + 4 (Using the **Commutative** Property of addition to change the order)
= (2s + 2s) + 4 (Using the **Associative** Property)
= 4s + 4 (Using the **Distributive** Property)

Note: 2s + 2s can be written as s(2 + 2) or 4s, using the Distributive Property to factor 2s + 2s. Textbooks often refer to terms like 2s and 2s as **"like terms"** because they have the same variable component. This permits factoring. For example, 3xy + 5xy = xy(3 + 5) = 8xy. In this example "3xy" and "5xy" are *like terms*. BUT, 3x + 5xy = x(3 + 5y) ≠ 8xy, because "3x" and "5xy" are not like terms.

6. Suppose a checking account contains $100 at the start of the week. Three checks are written during the week, one for $x and two for $(x + 1). Write an expression for the balance in the account at the end of the week in two ways:

The balance in dollars is 100 – x – 2(x + 1). This can be written as 100 – x – 2x – 2, or as 100 – 3x – 2 or as 98 – 3x, using properties of Real numbers to rewrite this expression in **equivalent** forms.
Note: the last part of the original expression is "-2(x + 1)." We are multiplying by –2, hence the "–2x – 2" in the expanded form.

*7. Write in **factored** form: 5x + 15 + 10z.*
Each of the terms of this **trinomial** have "5" as a factor. Therefore, we can use the **Distributive Property** to write this as 5(x + 3 + 2z).

8. Write in factored form: 5x + 10x^2.
Each of the terms of this **binomial** have "5" and "x" as factors. Therefore, using the Distributive Property, we can rewrite this as 5x(1 + 2x).

Symbol Manipulation

Combination of Expressions: Two or more symbolic expressions might be added (or subtracted or multiplied or divided) to make a third expression; or one expression might be **substituted** into another to create a third expression.

Factored form of an Expression: A quadratic trinomial of the form $ax^2 + bx + c$ can sometimes be expressed as a product of two binomial factors.
Note: Quadratics of the form $ax^2 + bx$ may also be factored.
Note: Finding factors of a quadratic trinomial is a somewhat trial and error procedure, using clues from the coefficients in the trinomial. Not all quadratic trinomials can be factored. It is important that students be able to recognize a quadratic trinomial, and know that some can be written in factored form and know what that factored format might look like. Knowing what each form of a quadratic tells you is important. However, since not all quadratic trinomials are factorable, students also learn other methods to *solve* quadratic equations.

Expanded Form of an Expression: Two binomials (or polynomials) can always be multiplied using the distributive property.

9. Suppose we know that tickets to a ball game cost $15 and a hot dog and drink will cost $10. Write an expression for the cost tickets for n people, cost of food for n people, and the total cost for n people.
Cost of tickets = 15n. Cost of food = 10n.
Total cost = 15n + 10n = $25n.

10. Suppose the ticket price per person is usually $x, but the ballpark will make a deal for groups of students, so that they pay $1 less per person for each person over 5 persons in the group. (This discount price applies to everyone in the group, up to a limit, of course). Write an expression for the price per person if n is greater than 5, and for the cost of taking n students to the ballpark.
Students often need to see information like this in a table so they can spot the pattern and write it in terms of a variable.

n	1	2	3	4	5	6	7
price pp	x	x	x	x	x	x – 1	x - 2

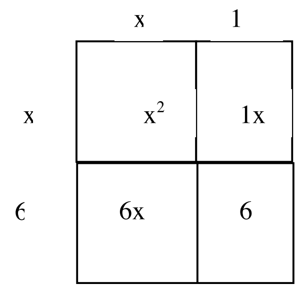
The price per person = x – (n – 5), if n > 5.

The total cost would be (n people)(price pp).
Substituting for the price per person,
total cost = n(x – (n – 5)), if n > 5.

11. Factor $x^2 + 5x + 6$.
The main tool that students have for factoring trinomials is an area model. Students also invent algorithms to make their area models more efficient. In this case we want a rectangle with binomials for length and width, so that when we multiply the length and width we get the quadratic trinomial $x^2 + 5x + 6$ for an area.

	x	2
x	x^2	2x
3	3x	6

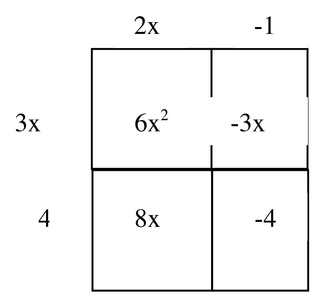
To make the procedure of finding the correct binomials for length and width more efficient students look at the constant term “6” and examine factor pairs. We could have 6 times 1 = 6, or 3 times 2 = 6. Trying (6, 1) as the constant terms in the binomials for the sides of the rectangle would give:

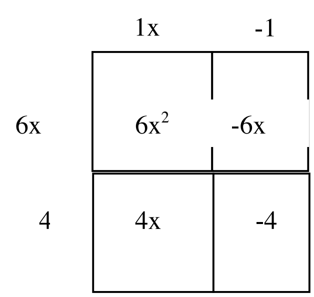


This arrangement gives $x^2 + 7x + 6$ for an area, *not* the required trinomial. The correct factor pair is (3, 2), as illustrated in the first sketch. Thus, $x^2 + 5x + 6 = (x + 3)(x + 2)$.

12. Factor $6x^2 – 5x – 4$.

This quadratic is more challenging to factor than the example in #11. This is because there are several possible factor pairs for the constant term (-4), *and* for the first term ($6x^2$). Below are two possibilities to check, using (-1, 4) for factors of –4, and (3x, 2x) and (6x, 1x) for $6x^2$:





In both cases the area has a term $6x^2$, and a constant term –4, but the trinomials produced are $6x^2 + 5x – 4$ and $6x^2 –2x – 4$ respectively. The first sketch is close to the target trinomial. We just have to have "-5x" in place of the "+5x." We can achieve this by changing the signs in the binomial factors.

	2x	1
3x	$6x^2$	3x
-4	-8x	-4

Thus, $6x^2 – 5x – 4 = (3x – 4)(2x + 1)$.

13. *Write in expanded form: (2x + 1)(x – 1)*
Students may use an area model to organize this multiplication. Thus,

	x	-1
2x	$2x^2$	-2x
1	1x	-1

$(2x + 1)(x – 1) = 2x^2 –2x +1x –1 = 2x^2 – x –1$.

Note: It is possible to use the Distributive Property without the aid of an area model.

$$\begin{aligned}(2x + 1)(x – 1) &= 2x(x – 1) +1(x – 1)\\ &= 2x^2 – 2x +1x – 1\\ &= 2x^2 –x – 1.\end{aligned}$$

The 4 terms that appear at the second last step are just the components of the area shown in the sketch.

One advantage of the area model as an organizer is that it extends to any number of terms multiplied by any number of terms. A

	disadvantage of the area model is that negative numbers are not very convincingly modeled.
Solve equations: • **Linear Equations** have already been a focus in *Moving Straight Ahead,* and *Thinking With Mathematical Models.* The Properties of Real numbers can now be used to solve linear equations that involve parentheses. • **Quadratic Equations: have a term with x^2 and** can be solved by using the factored form to write an equivalent equation, and then the zero product rule. The zero product rule states that if two numbers are multiplied to make zero then one or both of the numbers must be zero. **Note:** Not all quadratic expressions are factorable, and so "factoring" as a strategy has limited usefulness. However, if a quadratic equation has real solutions then "graphing" will always find these solutions. Another strategy for solving quadratic equations is the quadratic formula, which always finds solutions. The quadratic formula is not part of this unit of study.	*14. Solve the linear equation:* *$2 - 3(2x + 2) = 10x$.* $2 - 3(2x + 2) = 10x$ $2 - 6x - 6 = 10x$, using the Dist. Prop. $2 - 6 - 6x = 10x$, using the Comm. Prop. $-4 - 6x = 10x$ $-4 = 16x$. adding 6x to both sides of the equation. $X = -\frac{4}{16} = -0.25$ 15. *Solve the quadratic equation:* $x^2 + 2x - 3 = 0$. $x^2 + 2x - 3 = 0$ can be rewritten as $(x + 3)(x - 1) = 0$. (Check with an area model) Now we have a product of two factors equal to zero. So, it must be that $x + 3 = 0$, or $x - 1 = 0$. From $x + 3 = 0$ we have $x = -3$. From $x - 1 = 0$ we have $x = 1$. Therefore, there are two solutions, $x = -3$ and $x = 1$. Note: if we graph $y = x^2 + 2x - 3$ and then look for the points where $y = 0$ we have (-3, 0) and (1, 0). Thus, when $x = -3$, $0 = x^2 + 2x - 3$, and when $x = 1$, $0 = x^2 + 2x - 3$. Solutions: $x = -3$ and $x = 1$. 16. *Solve the quadratic equation: $6x^2 - 5x = 4$.* Since solving quadratic equations depends on using the zero product rule, we must first rewrite this equation as an equivalent equation: $6x^2 - 5x - 4 = 0$. Then write in factored form: $(3x - 4)(2x + 1) = 0$ (See example 12 above) Therefore, $3x - 4 = 0$ or $2x + 1 = 0$. From $3x - 4 = 0$ we have the solution $x = \frac{4}{3}$. From $2x + 1 = 0$ we have the solution $x = -\frac{1}{2}$. Note: this equation could also have been solved by graphing $y = 6x^2 - 5x - 4$ and looking for the x-intercepts.. *17. Solve the equation: $2x^2 - 5x = 0$* This is also a quadratic equation, though not a trinomial. It can be solved by factoring. $2x^2 - 5x = 0$

	$x(2x – 5) = 0$ $x = 0$ or $2x – 5 = 0$. $X = 0$ or $x = \frac{5}{2}$. Note: As in examples 16 and 17, this quadratic equation could also be solved by graphing $y = 2x^2 – 5x$ and looking or x-intercepts.
Parameters in equations of functions: Each of the function equations familiar to students has parameters (symbols) which relate to specifics in the context. These play particular roles in the pattern of change shown on the graph of the function. The general forms are as follows. • **Linear Function:** $Y = mx + b$, where the parameter “m” gives the rate at which y changes compared to x in the context, and also gives the slope of the graph. The parameter “b” gives the “starting” value for y, the value of y when $x = 0$, or the y-intercept on the graph. (See *Moving Straight Ahead*) • **Quadratic Function:** One form is $y = ax^2 + bx + c$, where “c” indicates the y-intercept on the graph, and “a” indicates whether the graph opens up or down and whether it is narrow or wide. Greater values of “a” indicate a faster rate of change. It is difficult to connect the parameters in this format to either the context or to the features of the graph. The factored format is $y = (ax + b)(cx + d)$. In this format we have the two x-intercepts $(-\frac{b}{a}, 0)$ and $(\frac{-d}{c}, 0)$. The line of symmetry lies half way between these x-intercepts. And the maximum or minimum value for y lies on this line of symmetry. The factored format connects productively to both graph and context. (See *Frogs and Fleas*) • **Exponential Function**: $y = ab^x$, where “a” gives the “starting” value for y, and “b” gives the growth factor. (See *Growing, Growing, Growing*)	*18. Identify whether the following equations are linear or quadratic functions, predict features of the graphs of the functions and explain what type of situation might be represented by such a function..* *a.* $y = (x +3)^2 – x^2$ *b.* $y = x(x + 3) + x^2$ a. $y = (x +3)^2 – x^2$ is equivalent to… $y = (x + 3)(x + 3) – x^2$ $y = x(x + 3) +3(x + 3) – x^2$ $y = x^2 + 3x + 3x + 9 – x^2$ $y = 6x + 9$. This is a linear function. The “m” = 6, which means that we expect the slope of the graph to be 6. This function might represent a situation in which one variable is growing at a rate of 6 units per unit change in another variable (for example, 6 mph or 6 cents per ounce or $6 per person). The “b” = 9, which means that the y-intercept is (0, 9). The function might represent a situation in which the starting value is 9 units (for example, 9 miles headstart or $9 cover charge). b. $y = -x(x - 3) - x^2$ is equivalent to …. $y = -x^2 + 3x - x^2$ $y = -2x^2 + 3x$. The “$-2x^2$” term tells us this is a quadratic function which opens down. In factored form this is equivalent to $y = x(-2x + 3)$. From the factored form we can tell that the x-intercepts on the graph of this function are the solutions of $0 = x(-2x + 3)$, which are $x = 0$ and $x = \frac{3}{2}$. The line of symmetry would be a vertical line midway between 0 and 1.5. Thus the line of symmetry is $x = 0.75$. The maximum y value will occur when $x = 0.75$. The maximum y value is (0.75, 1.125). This function might represent a situation in which one variable increases to a maximum of 1.125

	units and then decreases (for example, height of an arrow fired from a bow could be 1.125 meters after 0.75 seconds).