Thinking With Mathematical Models: Homework Examples from ACE

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<tr><td>ACE Investigation 1</td><td></td></tr>
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2.

The table shows the maximum weight a crane arm can lift at various distances from its cab. (See diagram in text.)

<table>
<tr><td>Dist (ft)</td><td>12</td><td>24</td><td>36</td><td>48</td><td>60</td></tr>
<tr><td>Weight (pounds)</td><td>7500</td><td>3750</td><td>2500</td><td>1875</td><td>1500</td></tr>
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a. Describe the relationship between distance and weight for the crane.

b. Make a graph of the (distance, weight) data. Explain how the graph's shape illustrates the relationship you described in part a.

c. Estimate the weight the crane can lift at distances of 18, 30, and 72 feet from the cab.

d. How, if at all, is the crane data similar to the data from the bridge experiments in Problems 1.1 and 1.2?

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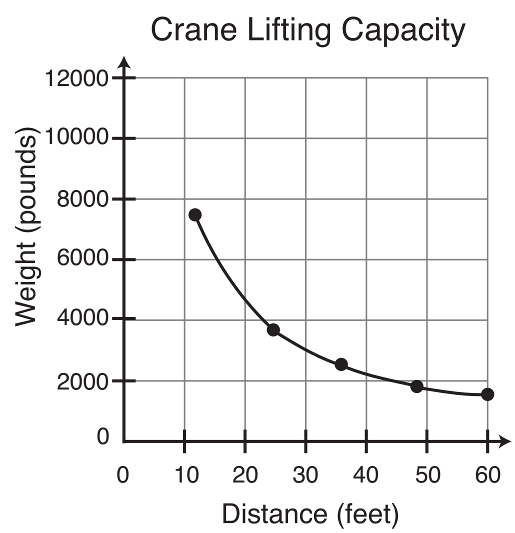
2.

a. As the distance from cab to weight increases, the weight decreases. But the rate of change is not constant. (see *Moving Straight Ahead*) For every 12 feet increase in the distance from the cab, the weight decreases, but not by the same amount every time. The weight decreases, but at a decreasing rate; that is the *change* in the weight is less and less every time.

b. The graph shows the weight decreasing as distance increases.



The curve of the graph shows that the initial decreases in weight are much larger than later decreases in weight.

c. 18 feet is half way between 12 and 24 feet. So the predicted weight for 18 feet should be between 7500 and 3750 pounds. If the weight decreased at a constant rate, the predicted weight would be 5625 pounds, exactly half way between 3750 and 7500 pounds. BUT the weight seems to be falling at a faster rate at the start, so the correct prediction is probably closer to 3750 than to 7500 pounds. At this point any prediction

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between 5625 and 3750 pounds would be sensible. (Later students will know more about this pattern, and be able to make better predictions.)

The same reasoning as above would put the predicted weight between 3750 and 2500 pounds, but closer to 2500 pounds.

This time we don't have collected data on either side of 72 feet. The predicted weight has to be less than 1500 pounds. Students might note that the weight decreased by 375 pounds from distance 48 to distance 60 feet. 72 feet is 12 feet more than 60 feet, so some might predict that the weight would be 1500 – 375 = 1125 pounds. Since the weight is decreasing, but at a decreasing rate, this prediction is too low.

d. The weight held by the bridge decreased as the length increased, but not at a constant rate.

Note: Students learn that this kind of relationship is called an *inverse proportional relationship*. This means that as the independent variable increases the dependent variable decreases, but not at a constant rate; in fact the product of the independent and dependent variables is a constant, for example, $xy = 10$ or, in general, $xy = a$ (which can also be written as $y = \frac{a}{x}$ etc.).

ACE Investigation 2

4.

This table gives average weights of purebred Chihuahuas from birth to age 16 weeks.

Age	0	2	4	6	8
Weight	4	9	13	17.5	21.5
Age	10	12	14	16	
Weight	25	30	34	39	

a. Graph the (age, weight) data, and draw a line that models the data pattern.

b. Write an equation in the form $y = mx + b$ for your line. Explain what the values of m and

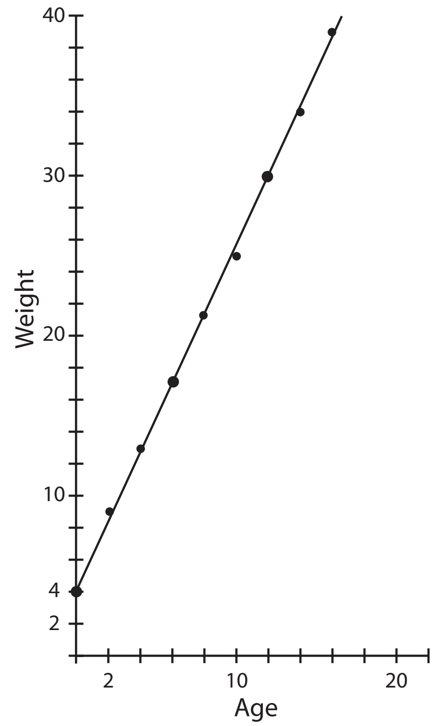
4.

a. When trying to decide where to draw a line that fits the data pattern, one wants to not let any one point be too influential. All points should "pull" on the line, so that the placement of the line reflects an overall trend. There should be about the same number of data points above as below the line. One tries to adjust the placement of the line so that the "gaps" between the line and the data points are minimized. The line may pass through several data points, or just a few points, or "miss" all points.

your line. Explain what the values of m and b tell you about this situation.

c. Use your equation to estimate the average weight of Chihuahuas for odd-numbered ages from 1 to 15 weeks.

d. What average height does your linear model predict for a Chihuahua that is 144 weeks old? Explain why this prediction is unlikely to be accurate.



b. The equation given here should fit whatever line is drawn in part a. From hand-drawn lines we can figure slope by reading two points that seem to be exactly on the line. In this case the line **might** pass through (6, 17.5) and (12, 30) making the slope = $\frac{12.5}{5}$ = 2.08.

Students might read the intercept from the graph (looks like approximately 5) or use the calculated slope to count back to the y-intercept. The "$y = mx + b$" equation produced should have the *calculated* slope and intercept in place of "*m*" and "*b*" respectively. The intercept tells us the average weight of a Chihuahua at age zero, that is at birth. The slope tells us how much an average Chihuahua is expected to grow each year. Student equations will vary but should be similar to W = 2.08A + 5.

c. Substituting age = 1 into the above equation we have W = 2.08(1) + 5= 7.08 ounces. The rest of the table is found by substituting appropriate values for age. Notice that these values will differ if a different line has been drawn in part a. However, the weight values found by students should be similar to those shown.

A	1	3	5	7	9
W	7.08	11.24	15.4	19.56	23.72

d. Substituting $A = 144$ into the above equation, we have $W = 2.08(144) + 5 = 304.52$ ounces or about 19 pounds. This is unreasonably heavy, not a good representation of an "average" weight of a Chihuahua that is 144 weeks (nearly 3 years) old. This illustrates that mathematical models, or in this case a line of best fit, can not be trusted to continue to model the data well when we stray too far from the given data.

26. The following formulas give the fare f in dollars that two bus companies charge for trips of d miles.

Transcontinental: $f = 0.15d + 12$

Intercity Express: $f = 5 + 0.20d$.

In parts a to c use a graph to estimate the answer. Then find the answer by writing and solving an equation or inequality.

a. For Transcontinental, how many miles is a trip that costs $99?
b. For Intercity Express, how far can a person travel fro a fare that is at most $99?
c. Is there a distance for which the fare for the two bus lines is the same? If so give the distance and the fare?

26.

a. Students may use their calculators to graph $y = 0.15x + 12$ (or do this by hand).
Tracing along the graph to the point (?, 99) we find (580, 99).
(Note: students may not be able to have the trace feature on their calculators read the point where y is exactly 99. They can use the table feature.)
We can also find the answer using symbol manipulation by setting up
$99 = 0.15d + 12$ and solving for d.
We get $87 = 0.15d$, so $d = 580$ miles.

b. Tracing along a graph of $y = 5 + 0.20x$ we find approximately (470, 100).
We can also find the answer by solving
$99 = 5 + 0.20d$.
We get $94 = 0.20d$, so $d = 470$ miles. The person can travel any distance UP TO 470 miles for this fare.

c. We can solve this problem two ways. Each of the above graphs shows ALL possible combinations of distance and cost for each company. So **the point of intersection** shows a combination that works for **both** companies. This (140, 33).

OR we can **solve this symbolically** by setting up and solving the equation:
0.15*d* + 12 = 5 + 0.20*d*.
7 = 0.05*d*.
d = 140 miles. (Check: Transcontinental would charge 0.15(140) + 12 or $33. Intercity would charge 5 + 0.2(140) or $33.)

ACE Investigation 3

4 – 7.
For each of the following tables, determine whether the relationship between *x* and *y* is an **inverse variation.** If it is, write an equation that expresses the relationship.

4.

X	1	2	3	4	5	6
Y	10	9	8	7	6	5

5.

X	1	2	3	4	5	6
Y	48	24	16	12	9.6	8

6.

X	2	3	5	8	10	15
Y	50	33	20	12.5	10	6.7

7.

X	0	1	2	3	4	5
Y	100	81	64	49	36	25

4 - 7.
In every table we see that as ***x* increases *y* decreases.** This is ONE of the conditions for an inverse variation pattern. By itself this is not enough to identify an inverse variation. Students may also make graphs to see if the characteristic shape of an inverse variation relationship is present. **For inverse variation the variables must fit the equation *xy* = *a*, for some constant *a*, where *a* is non-zero..**

4. The rate of change of y is **constant**: rate = $\frac{-1}{1}$.
   This is a **linear** pattern.

5. We see that as *x* increases, *y* decreases but the **rate of decrease is slowing down.** This is characteristic of an inverse variation pattern. We need to check the last characteristic of inverse variation patterns: is the product of variables a constant? In this case ***xy* = 48** for every pair of (*x, y*) values. This is an **inverse variation.** (If we graphed these points the **characteristic curved shape of the graph** of an inverse variation relation would appear.)

6. Since the *x* values are not changing by regular increments we cannot see easily whether the rate is constant or not. If we compare the difference in y to the difference in *x* on each

	interval we will see that the rate is NOT constant, so this is not a linear relationship. Since every pair of values fits the equation ***xy* = 100** this is also an **inverse variation**. (3, 33) does not fit exactly; perhaps this is a rounding error. But the point (3, 33) would lie very close to the graph of $xy = 100$, or $y = \frac{100}{x}$. 7. If we graphed these points the graph would look very like the characteristic curved shape of an inverse variation relationship, BUT there is no constant value for the product of the (*x, y*) values. In fact, the existence of a y-intercept should alert us to that. (0)(100) = 0. None of the other (*x, y*) products are zero. Note: there will be **no y-intercept** on the graph of an inverse variation relationship. This is because neither $x(0) = a$, nor $(0)y = a$ could have a solution.
28. Jamar takes a 10-point history quiz each week. Here are his scores on the first 5 quizzes.: 8, 9, 6, 7, 10. a. Jamar misses the next quiz and gets a 0. What is his average after 6 quizzes? b. After 20 quizzes, Jamar's average is 8. He gets a 0 on the 21st quiz. What is his average after 21 quizzes? c. Why did a score of 0 have a different effect on the average when it was the 6th score than when it was the 21st score?	28. a. Jamar had a total of 40 points after 5 quizzes. (An average of 8.) This total is unchanged after 6 quizzes, but now he has to divide by 6 to get his average score. Average = $\frac{40}{6}$ = 6.7 approx. b. Jamar must have accumulated a total of 160 points over 20 quizzes to average 8 per quiz. So, with the same total over 21 quizzes the average drops to $A = \frac{160}{21} = 7.6$. c. Students might argue that missing a quiz is like losing a potential 10 points. Spreading a loss of up to 10 points over 21 quizzes will have less effect than spreading the loss of 10 points over 6 quizzes. That is the value of $\frac{10}{n}$ will decrease as *n* increases. $\frac{10}{6}$ per quiz is a greater loss than $\frac{10}{21}$ per quiz. Another way to think about this is to write an equation for the average after *n* quizzes, when the average after *n* - 1 quizzes was 8. If the

average after n - 1 quizzes was 8 then the total number of points is $8(n-1)$. If this $8(n-1)$ points remains unchanged because the next quiz is a 0, then the average after n quizzes is $\frac{8(n-1)}{n}$. The graph or table of the relationship A = $\frac{8(n-1)}{n}$ show that as n increases so does A, but at a slower and slower rate, and the old average of 8 is never regained.

n	2	3	4	5	6	7
A	$\frac{8}{2}$	$\frac{16}{3}$	$\frac{24}{4}$	$\frac{32}{5}$	$\frac{40}{6}$	$\frac{48}{7}$

We see that $\frac{48}{7} > \frac{40}{6} > \frac{32}{5}$ etc. And all of these fractions are less than 8.