Frogs, Fleas and Painted Cubes: Homework Examples from ACE

Investigation 1: *Introduction to Quadratic Functions,* ACE #7, 14
Investigation 2: *Quadratic Expressions,* ACE #17, 24, 26, 36, 51
Investigation 3: *Quadratic Patterns of Change,* ACE #3, 11, 19, 27
Investigation 4: *Frogs Meet Fleas on a Cube: More Applications of Quadratic Functions,* ACE #5

Investigation 1: *Introduction to Quadratic Functions*
ACE #7

The equation for the areas of rectangles with a certain fixed perimeter is $A = l(20 - l)$ where l is the length in meters.

a. Describe the graph of this equation.
b. What is the maximum area for a rectangle with this perimeter? What dimensions correspond to this area? Explain.
c. A rectangle with this perimeter has a length of 15 meters. What is its area?
d. Describe 2 ways you can find the perimeter. What is the perimeter?

In Investigation 1 students connected the equation for the areas of rectangles with fixed perimeters of 20 to the graph of the equation and to the table. They should recognize the format of the equation in this ACE question, and may be able to predict the shape of the graph and the fixed perimeter from the symbols. If they are not able to do this yet they can still make a table from the equation, and proceed from there to a graph.

a. The table would be:

Length	0	1	2	3	4
Area	0	1(19) = 19	2(18) =36	3(17) = 51	4(16) = 64
Length	5	6	7	8	9
Area	5(15) = 75	6(14) = 84	7(13) = 91	8(12) = 96	9(11) = 99
Length	10	11	12	13	etc
Area	10(10) = 100	11(9) = 99	12(8) = 96	13(7) = 91	etc

From the table we can predict that the graph will be a curve (not a constant slope) that will rise to a maximum when the length is 10. This U shaped curve or parabola will cross the x-axis twice, at
(0, 0) and again at (20,0). (This second intercept has not yet appeared in the table above.)

b. The maximum area occurs when length and width are both 10 meters. This area is 100 square meters. (See table.)

c. Students can either substitute $l = 15$ into the equation or use the table.
$A = 15(20 - 15) = 75$ square meters. Notice that this rectangle has the same area as a rectangle with length 5 meters.

d. Students could find the fixed perimeter by using clues from the table or graph. For example, they might say that a rectangle with length 15 meters has area 75 square meters, and so the width must be $^{75}/_{15}$ = 5 meters. Therefore, the perimeter is 15 + 5 + 15 + 5 = 40 meters. Or they might use the format of the equation to deduce the length and width. If $A = l(20 – l)$ where l is the length, then the width must be the "$20 – l$" factor. So, if length is 4 meters, for example, then width must be 20 – 4 = 16 meters, and perimeter must be 4 + 16 + 4 + 16 = 40 meters. Or, generally, perimeter = length + width + length + width = $l + 20 – l + l + 20 – l$ = 40 meters.

Investigation 1: *Introduction to Quadratic Functions*
ACE #14

The equation $p = d(100 – d)$ gives the monthly profit p a photographer will earn if she charges d dollars for each print.

a. Make a table and a graph for this equation.
b. Estimate the price that will produce the maximum profit. Explain.
c. How are the table and graph for this situation similar to those you made in Problem 1.1? How are they different?

Students should recognize the format of this equation, even though the context has changed from areas of rectangles with fixed perimeters to profit. They should be able to predict that the graph of this relationship will be a parabola. They may be able to predict where the maximum profit will occur, by examining the symbols.

a. Table *started* here. Not completed.

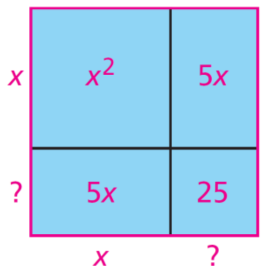
Dollars per print	0	1	2	3	4
Profit	0	1(99) = 99	2(98) = 196	3(97) = 291	4(96) = 384

b. By comparing this equation to the equation for areas of rectangles with a fixed perimeter of 200, A = l(100 – l), students should expect that entries for profit in the table will rise to a maximum and then repeat. For example, we have a repeated profit value of 900, when d = 10 and later when d = 90. Knowing where the entries for profit *begin* to repeat tells students where the maximum occurs. Not completed here.

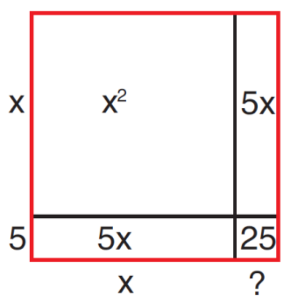
c. Not completed here.

Investigation 2: *Quadratic Expressions*
ACE #17

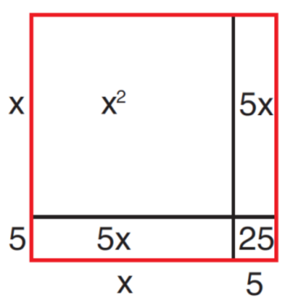
Write two expressions, one in factored form and one in expanded form, for the area of the rectangle outlined in red.



The area of the red rectangle is made of several parts. Each of these smaller areas is known, and for each of the smaller areas we have a clue about one of the dimensions. For the rectangle in the lower left corner labeled "5x" we know that one side has length x. Thus, the other side must have length 5. The sketch below indicates this additional piece of information.



The rectangle in the lower right corner has area 25. We know from the above sketch that one side has length 5. Thus, the other side must also have length 5. This completes all needed information, as shown below.

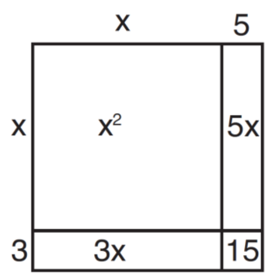


Now we can see that the area of the red rectangle can be written as $A = x^2 + 5x + 5x + 25$, which is the expanded form. But it can also be written as $A = L \times W = (x + 5)(x + 5)$, which is the factored form.

Investigation 2: *Quadratic Expressions*
ACE #24

Use the Distributive Property to write the expression in expanded form.
$(x + 3)(x + 5)$

Students can find the expanded form of this expression by making a sketch of a rectangle with length $x + 3$ and width $x + 5$.



However, they should also be able to use the Distributive Property directly without the aid of a diagram, as follows:

$$(x + 3)(x + 5) = (x + 3)x + (x + 3)5$$
$$= x^2 + 3x + 5x + 15$$
$$= x^2 + 8x + 15.$$

Notice that the terms in this expanded expression are just the parts of the area in the rectangular model.

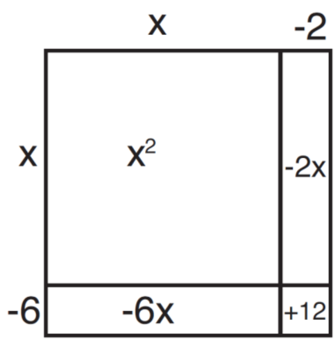
Investigation 2: *Quadratic Expressions*
ACE #26

Use the Distributive Property to write the expression in expanded form.
$(x – 2)(x – 6)$.

$(x – 2)(x - 6) = (x – 2)x – (x – 2)6$
$= x^2 – 2x – 6x$ ……. Not completed here.

(Note: This question is challenging because of the negative signs. To help you think about how to complete the problem it might help to compare this to
$(x – 2)(x + 6) = (x – 2)x + (x – 2)6$
$= x^2 – 2x + 6x - 12$)

Note: drawing a rectangular model is not so convincing here because of the negative terms involved in the two factors. But, if we just use the model as an organizer we can see the correct 4 terms in the expanded format.



Investigation 2: *Quadratic Expressions*
ACE #36

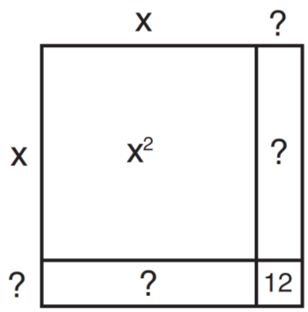
Write each expression in factored form.

a. $x^2 + 13x + 12$.

b. $x^2 - 13x + 12$

c. Etc.

a. Students might try to make a rectangle with this, using the clues in the "x^2" term and the "12" term to find the length and width of the rectangle.



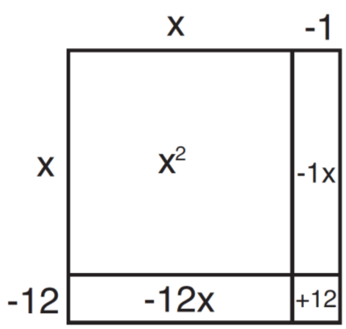
Or they may try to use the Distributive Property as follows:

$x^2 + 13x + 12 = x^2 + 12x + 1x + 12$

$= x(x + 12) + 1(x + 12)$

$= (x + 12)(x + 1)$.

b. As above, the area model would be:



Notice that the negative terms on the dimensions may make this strategy less convincing.

Or, using the Distributive Property:

$x^2 - 13x + 12 = x^2 - 1x - 12x + 12$

$= x(x - 1) - 12(x - 1)$

$= (x - 1)(\ldots..)$ Not completed.

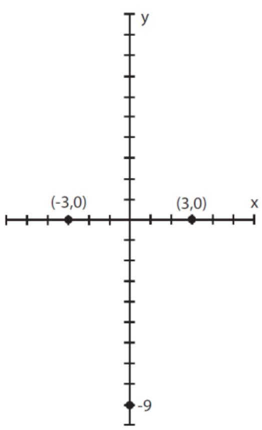
Investigation 2: *Quadratic Expressions*
ACE #51

Rewrite each equation in expanded form. Then, give the x- and y-intercepts, the coordinates of the maximum or minimum point, and the line of symmetry for the graph of each equation.

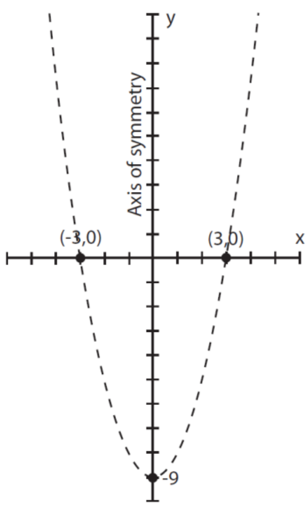
a. $y = (x - 3)(x + 3)$
b. $x(x + 5)$
c. $(x + 3)(x + 5)$
d. etc.

a. All of this information can be gleaned from the equation, without resorting to making the actual graph. The expanded form for the equation $y = (x - 3)(x + 3)$ is $x^2 - 3x + 3x - 9$, some students might simplify this as $x^2 - 9$. Using the equation $y = (x - 3)(x + 3)$, there are two ways that the y value can be zero: when $x = 3$ and when $x = -3$. If $x = 3$ then $y = (3 - 3)(3 + 3) = 0$. So (3, 0) is an x-intercept. Likewise (-3, 0) is an intercept. (Students should recall Problem 2.1, where they investigated and graphed $y = (x + 2)(x - 2)$.)

The y-intercept occurs when $x = 0$, so $y = (0 - 3)(0 + 3) = -9$.
Thus, the y-intercept is (0, -9).
This gives us 3 points on the curve, as shown below:

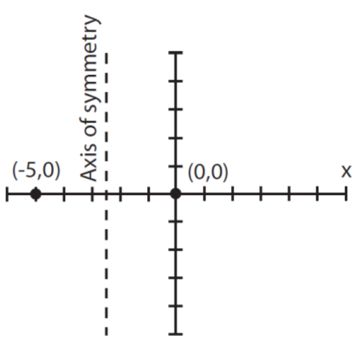


The line of symmetry is a vertical line which passes through a point midway between the two x- intercepts, that is, midway between (-3, 0) and (3, 0). Thus, the line of symmetry passes through (0, 0). We use an equation, $x = 0$, to describe this vertical line.



The minimum point lies on this axis of symmetry. Thus the minimum point must have $x = 0$ for the x-coordinate. The other clue we have is that the minimum point is ON the curve, and so the coordinates must satisfy $y = (x – 3)(x + 3)$. Substituting $x = 0$, we have $y = (0 – 3)(0 + 3) = -9$. The minimum point is (0, -9).

b. The expanded form for $x(x + 5)$ is $x^2 + 5x$. As in part a, we find the x-intercepts by considering what x values would make $x(x + 5) = 0$. These are $x = 0$ and $x = -5$. The x-intercepts are (0, 0) and (-5, 0). The y-intercept has an x-coordinate of 0, so $y = 0(0 + 5) = 0$. So the y-intercept is (0, 0). The axis of symmetry passes through a point midway between the x-intercepts, (0, 0) and (-5, 0). This point is (-2.5, 0). The axis of symmetry has the equation $x = -2.5$.



The minimum point has $x = -2.5$ for an x- coordinate. So, $y = -2.5(-2.5 + 5) = 6.25$. The minimum point is (-2.5, 6.25).

c. Etc.

Investigation 3: *Quadratic Patterns of Change*
ACE #3

In Problem 3.1, you looked at triangular numbers.

a. What is the 18^{th} triangular number?
b. Is 210 a triangular number? Explain.

In Problem 3.1, students found an equation for any triangular number. Depending on how they saw the pattern in these numbers they may have the equation:

$$t = \frac{n^2}{2} + \frac{n}{2} \quad \text{or} \quad t = \frac{(n+1)^2 - (n+1)}{2}$$

a. Whatever equation they found in Problem 3.1, they can now apply this using 18 for n. T = 0.5(18)(18 + 1) = 171.

Or they could make a table and extend the pattern they see.

N	1	2	3	4	5	6	7	…	18
T	1	3	6	10	15	21	28	…	171

b. Students can make a table with triangular numbers as above, and extend this table to see if 210 fits the pattern in the table.

Or they can try to find out if there is an n value that fits the equation 210 = 0.5n(n + 1). The first thing they might try is doubling both sides of the equation to get 420 = n(n + 1). Now they are looking for two consecutive numbers that multiply to make 420. Since 20 x 21 = 420, n must be 20.

Investigation 3: *Quadratic Patterns of Change*
ACE #11

A company rents five offices in a building. There is a cable connecting each pair of offices.

a. How many cables are there in all?
b. Suppose the company rents two more offices. How many cables will they need in all?
c. Compare this situation with Case 3 in Problem 3.2.

a. Students might think about this in different ways. For example, they might say that each office has to be connected to 4 other offices. This might make them think that the number of cables must be 5 x 4. But this double counts each cable; the cable connecting office A to office B is not different from the cable connecting office B to office A.

   Or, they might call the offices A, B, C, D and E, and then list the connections needed: AB, AC, AD, AE, BC, BD, BE, CD, CE, DE. This is 10 connections.

   Or they might make a sketch and notice that A has to be connected to 4 offices, B to 3 offices, C to 2 offices and D to 1 office, making 4 + 3 + 2 + 1 = 10 offices.

   Or, they might notice that this is essentially the same problem as 5 people shaking hands or high- fiving each other, and use their equation from Problem 3.2C.

b. Whichever strategy students used in part a they can now apply the same strategy to the situation with 7 offices. Not completed here.

c. Not completed here.

Investigation 3: *Quadratic Patterns of Change*
ACE #19

The table represents a quadratic relationship. Copy and complete the table.

X	0	1	2	3	4	5	6
Y	0	4	6	6	?	?	?

Quadratic relationships have a specific pattern in the rate of change in the dependent variable, y. This rate of change in y can be seen in the table or on the graph of the parabola. The y-values change at different rates depending on where on the parabola you track the change. Students may have formalized this idea in a discussion in class; or they may not have formalized their ideas about the pattern of change in the y-values, and they may have to rely on the idea of symmetry in the table and the parabola, in order to complete the table. Making a graph of the given points will help them see how to preserve the pattern of change and the symmetry.

(Graph not shown here.)

If we find the differences in the y-values as the x- values increase by 1 unit, we do not find a constant difference in y; but if we find the differences of the differences in y (called the "second differences") then these are constant. Thus, looking for a pattern in the differences in the y-values is key to identifying a quadratic relationship. The "first differences" and "second differences" are written into the table below as first steps in identifying the pattern of change in y values.

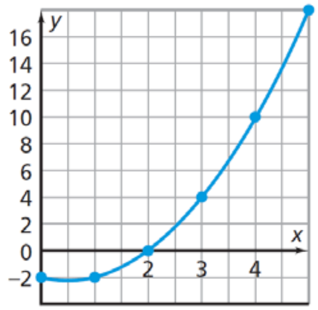
X	0	1	2	3	4	5	6
Y	0	4	6	6	?	?	?
1st Diff	4	2	0	?	?	?	
2nd Diff		-2	-2				

Now we can see how to choose y values which preserve the pattern of change in the first differences and make all the second differences -2. We can also see how to choose y-values that will preserve the symmetry in the parabola associated with this table. The repeated y-values will appear as points which are mirror images of each other in the axis of symmetry x = 2.5.

X	0	1	2	3	4	5	6
Y	0	4	6	6	4	0	-6
1st Diff	4	2	0	-2	-4	-6	
2nd Diff		-2	-2	-2	-2	-2	

Investigation 3: *Quadratic Patterns of Change*
ACE #27

The graph shows a quadratic function. Extend the graph to show x-values from -4 to 0.



We can extend this graph visually or by making a table and extending the numerical pattern we see in the table.

X	-4	-3	-2	-1	0	1	2	3	4
Y	?	?	?	?	-2	-2	0	4	10
1st diff	-8	-6	-4	-2	0	2	4	6	
2nd diff		2	2	2	2	2	2	2	

In order to preserve the pattern of differences in the table we need to extend the table or graph to include the points (-1, 0), (-2, 4), (-3, 10), (-4, 18).

Investigation 4: *Frogs Meet Fleas on a Cube: More Applications of Quadratic Functions*
ACE #5

Kelsey jumps from a diving board, springing up into the air and then dropping feet-first. The distance d in feet from her feet to the pool's surface t seconds after she jumps is
$d = -16t^2 + 18t + 10$.

a. What is the maximum height of Kelsey's feet during this jump?
When does the maximum height occur?

b. When do Kelsey's feet hit the water?

c. What does the constant term 10 in the equation tell you about Kelsey's jump?

a. Students can use their graphing calculators to make a graph or a table, so that they can see where the maximum height occurs. The table will look like:

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
d	10	11.64	12.96	13.96	14.64	15	15.04	14.76

From this table (or by inspecting the graph) we can deduce that the maximum occurs at approximately
0.6 seconds, and that the maximum height is approximately 15.04 feet. Because the values for d are not arranged symmetrically in the table we can be sure that we don't have the exact maximum (t, d) point in the table. If we want more accuracy we can have the calculator produce another table with even smaller increments in t.

t	0.55	0.56	0.57	0.58	0.59	0.60
d	15.06	15.062	15.062	15.058	15.05	15.04

From the second table we can deduce that the maximum height is about 15.062 feet and occurs at 0.565 seconds.

b. This question asks when the height is 0 feet.
Students can answer this by extending the table. Not completed here.

c. When you substitute t = 0 into the equation $d = -16t^2 + 18t + 10$, you get d = 10.
This is the height of Kelsey's feet above the surface of the water at t = 0, that is at the start of the jump. So the diving board must be 10 feet above the surface of the water.