Mathematics Background

The Language of Number Theory

Prime Time addresses the basics of number theory: factors, multiples, prime and composite numbers, even and odd numbers, square numbers, greatest common factors, and least common multiples.

The concepts of factor and multiple are interdependent:

If A is a **factor** of B, then B is a **multiple** of A.

This means that we can find a number C such that the product of A and C equals B, that is, A × C = B. From this we see that factors always come in pairs. For example, we know that 3 × 4 = 12. This says that 3 is a factor of 12 and that 4 is a factor of 12. The two are a **factor pair** because their product is equal to 12. Here are several statements that describe the relationships in the number fact 3 × 4 = 12:

3 × 4 = 12

3 is a factor of 12.
3 is a divisor of 12.

4 is a factor of 12.
4 is a divisor of 12.

12 is the product of 3 and 4.
12 is a multiple of 3.
12 is a multiple of 4.
12 is divisible by 3.
12 is divisible by 4.

Whole numbers can be classified by the type and number of factors they have. For example:

- **Even numbers** have a factor of 2 or are divisible by 2.
- **Odd numbers** do not have a factor of 2 or are not divisible by 2.
- **Prime numbers** have exactly two factors, 1 and itself.
- A **composite number** is a number that is not prime.

It is important that students learn to use the language described above.

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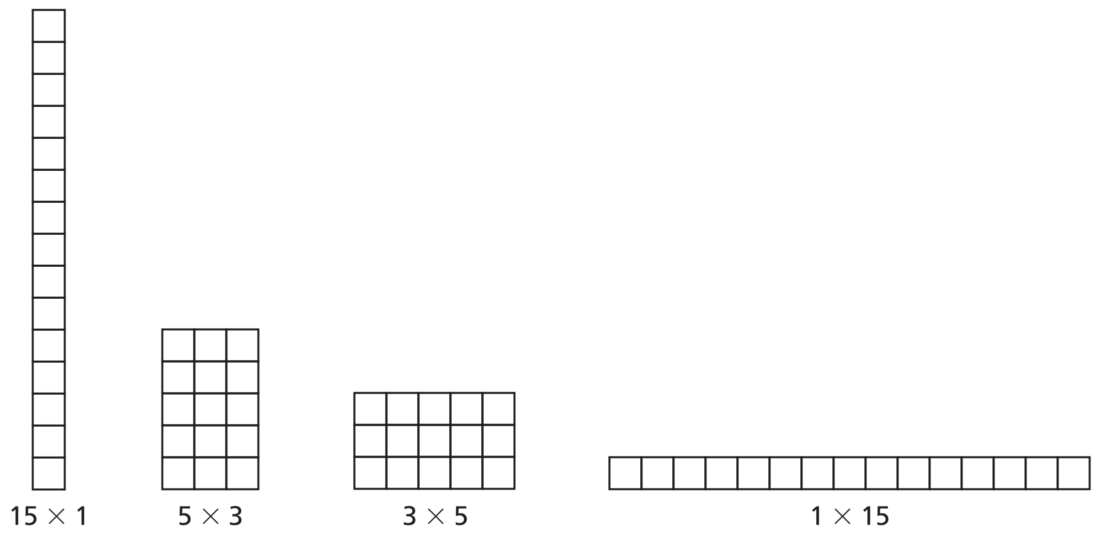
Factor Pairs and the Square Root of a Number

In Investigation 1, students find all the factors of a number by listing. They find all the factor pairs of a number by finding all the rectangles that can be formed from unit tiles, as shown for the number 15. The factors of 15 are 1, 3, 5, and 15.



Some numbers only have one possible rectangle (disregarding orientation). These numbers are prime numbers like 2, 3, 5, 7, etc. Some numbers, like 4, 9, 16, etc., can have a rectangle that is a square. These numbers are called **square numbers.**

An important question arises naturally out of the investigation of rectangles one can make from a fixed number of tiles:

> *How do I know when I have all of the possible rectangles for a given number of tiles, excluding orientation variations?*

Another form of the question is

> *When listing all of the factor pairs of a number, how can I predict the point where the factor pairs begin to repeat in reverse order?*

A more sophisticated version of the question is

> *What numbers do I have to check to find all the factors of anumber or to show that the number is prime?*

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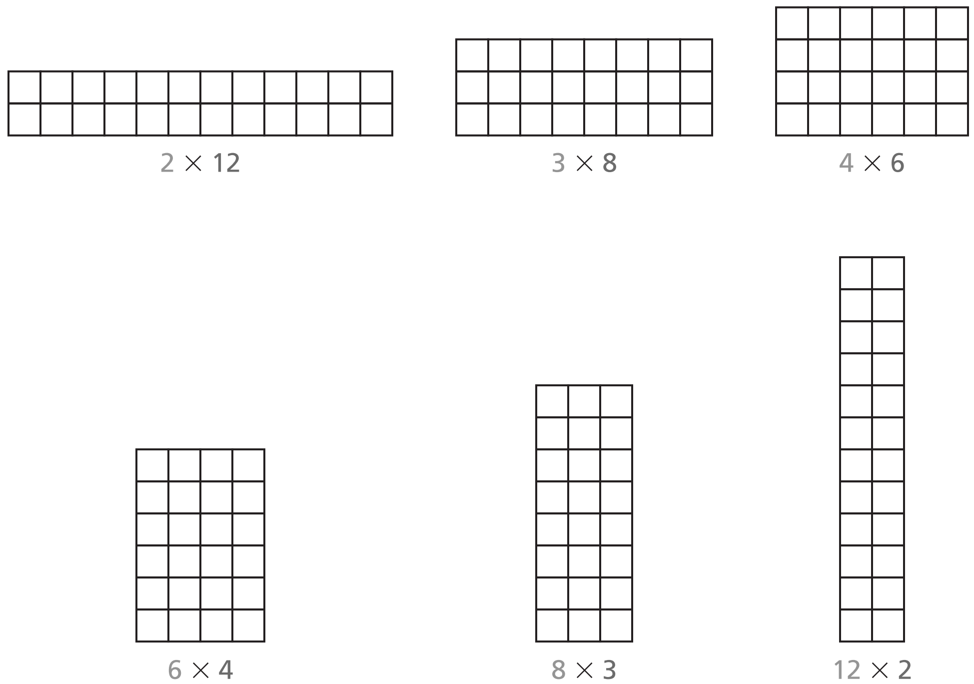
The key to finding all the factors of a given number, *n*, is to examine systematically the whole numbers that are less than or equal to the square root of *n*. Your students probably have little understanding of square roots at this stage. They are more likely to understand it as the point where the factors in the factor pairs reverse order so that the first factor is greater than the second.

Factor Pairs of 24

1, 24
2, 12
3, 8
4, 6
6, 4
8, 3
12, 2
24, 1

Here, the factor pairs reverse order.

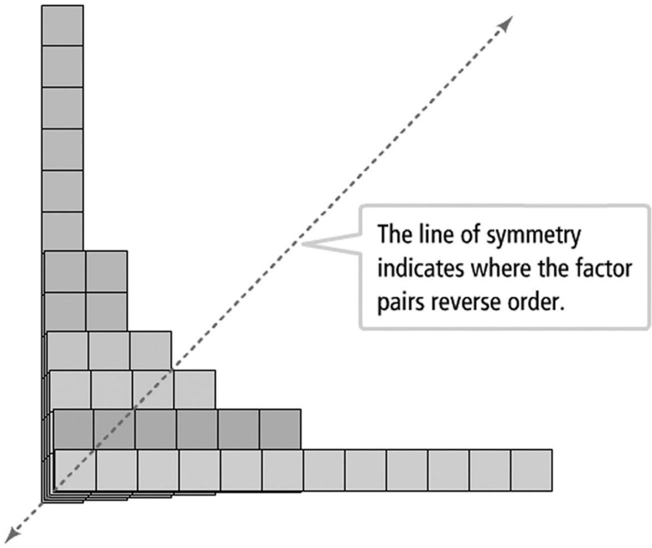
When forming the rectangles for 24, students observe that the 4 × 6 (or 6 × 4) rectangle is the most squarelike of the rectangles. They also notice that as one edge of a rectangle for 24 gets longer, the other edge gets shorter.



At some point, the numbers on the left in an ordered list of pairs become greater than the numbers on the right.

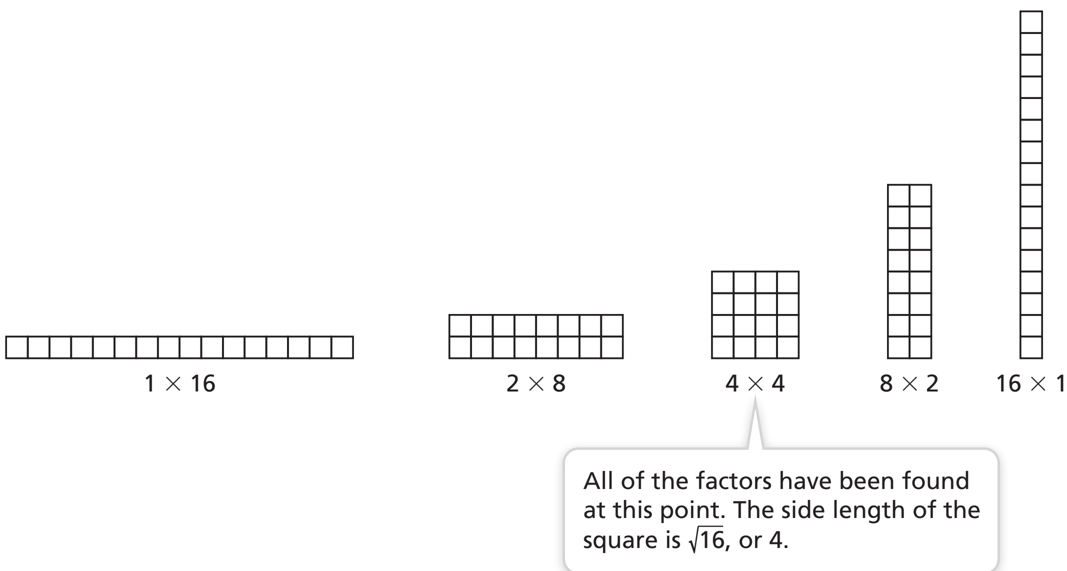
Students can also see where the factor pairs reverse order geometrically. By superimposing the rectangles for a number on top of each other, students can see the symmetry of the factor pairs. Visit *Teacher Place* at mathdashboard.com/cmp3 to see the complete video.

Factor Pairs of 12



When the pattern of rectangles that form factor pairs gets as close to a square as possible, we have found all the factors of the given number. When the given number is a square number, we have found all of the factors once you reach the rectangle that is a square. The length of the side of that square is the square root of the original number. For example, consider the number 16.

Example



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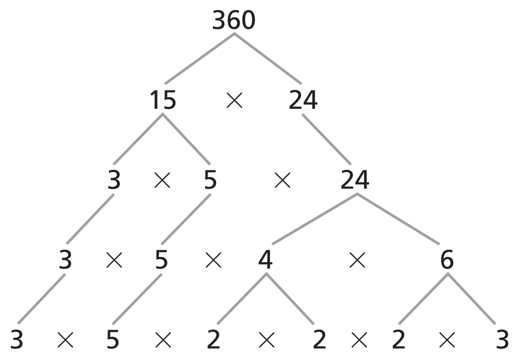
Knowing that we have found all of the factor pairs ultimately depends on understanding that for any number, the two factors in a factor pair of a given number lie on opposite sides of the square root of that number. For example, analyze the factors and their pairings for 30 and 36. If you draw lines connecting the factor pairs in each, you get the following:



$$360 = 2^3 \times 3^2 \times 5$$

You can see that the factor pairs arrange themselves around the square root of each number. We expect that students will see that the pair of factors that make the most squarelike rectangle indicates the place where the factors in the factor pairs reverse order and repeat.

Later in the Unit when students are finding the prime factorization of numbers, they will know that they have found all of the prime factors of a number once they have reached the square root of the number. They will have found all of the factor pairs involving a prime once they have checked all the prime numbers less than the square root. If they find no such prime factors, then they can conclude that the number is prime. Note that square roots are not formally developed until *Looking for Pythagoras* in the eighth grade.

Identity Properties

Students observe that 1 is a factor of every whole number. The product of 1 and another number, A, is A; that is, $1 \times A = A$. Hence, we call the number 1 the **multiplicative identity**. For similar reasons, 0 is the **additive identity**. Zero plus any whole number equals the whole number; that is, $0 + A = A$.

Identity Properties

Algebra	Example
$A + 0 = 0 + A = \boldsymbol{A}$	$6 + 0 = 0 + 6 = \boldsymbol{6}$
$A \times 1 = 1 \times A = A$	$6 \times 1 = 1 \times 6 = 6$

These ideas are useful when discussing multiplying and dividing fractions and will be discussed in later Connected Mathematics Units, including *Decimal Operations* in the sixth grade and *Accentuate the Negative* in the seventh grade.

Even and Odd Numbers

In Investigation 4, students are asked to make conjectures about the sum and product of two whole numbers. They predict whether the sum or product is even or odd. Students use square tiles or numeric reasoning to justify their conjectures.

A formal proof that the sum of two even numbers is even is given.

> **Proof**
> Any even number y can be written as $y = 2n$. Let's take two even numbers, $y = 2n$ and $z = 2m$. Then $y + z = 2n + 2m = 2(n + m)$, which is even.

The proof that the sum of two odd numbers is even is similar.

> **Proof**
> Any odd number a can be written as $a = 2z + 1$. Let's take two odd numbers, $a = 2z + 1$ and $b = 2w + 1$. Then $a + b = (2z + 1) + (2w + 1)$. Then $2z + 2w + 2 = 2(z + w + 1)$, which is even.

Similar proofs work for the sums of even and odd numbers and the products of even and odd numbers. These formal proofs are done in the eighth-grade Unit *Say It With Symbols*.

Classifying a Number by the Sum of Its Proper Factors

In the first Investigation, students play the Factor Game. This game provides an opportunity to assess students' understandings of factors and multiples.

The Factor Game Directions

1. Player A chooses a number on the game board and circles it.
2. Using a different color, Player B circles all the proper factors of Player A's number.
3. Player B circles a new number, and Player A circles all of the factors of the new number that are not already circled.
4. The players take turns choosing numbers and circling factors.
5. If a player chooses a number with no uncircled factors, that player loses their current turn and scores no points.
6. The game ends when there are no numbers left with uncircled factors.
7. Each player adds the numbers circled with his or her color. The player with the greater total wins.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

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A natural outcome after playing the Factor Game and analyzing "best first moves" is to analyze numbers by characteristics of their factors. Students will learn about abundant, deficient, perfect, and near-perfect numbers in the *Extensions* of Investigation 1.

Term	Definition	Example
prime number	A number with exactly two factors, 1 and the number itself.	29; the factors of 29 are 1 and 29. Note that 29 is the best first move because it has only 1 as a proper factor. The opponent only gets only one point.
abundant number	A number for which the sum of all its proper factors is greater than the number itself.	24; the sum of the proper factors of 24 is $1 + 2 + 3 + 4 + 6 + 8 + 12 = 36$, which is greater than 24.
deficient number	A number for which the sum of all its proper factors is less than the number itself.	21; the sum of the proper factors of 21 is $1 + 3 + 7 = 11$, which is less than 21.
perfect number	A number for which the sum of all its proper factors is equal to the number itself.	6; the sum of the proper factors of 6 is $1 + 2 + 3 = 6$, which is equal to 6. Note that 6 and 28 are the only perfect numbers between 1 and 30.
near-perfect number	A number for which the sum of all its proper factors is one less than the number itself.	16; the sum of the proper factors of 16 is $1 + 2 + 4 + 8 = 15$, which is one less than 16. Note that 16 is also deficient.

Near-perfect numbers are useful for finding perfect numbers. Euclid discovered this method:

1. Start with a near-perfect number whose proper factors have a prime sum.
2. Multiply the sum of the factors by the greatest power of 2 less than the sum. The product will be a perfect number.

Example

The number 4 is near-perfect, and the sum of its proper factors is 3, which is prime. The greatest power of 2 less than 3 is 2, and $3 \times 2 = 6$, which is perfect.

The number 8 is also near-perfect, and the sum of its proper factors is 7, which is prime. The greatest power of 2 less than 7 is 4, and $7 \times 4 = 28$, which is perfect.

Euclid's method will not work for the near-perfect number 16 because the sum of its proper factors is 15, which is not prime.

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Euclid's method always produces even perfect numbers. No one knows whether there are any odd perfect numbers, but we do know that powers of 2 (e.g., 2, 4, 8, 16, 32,. . . 2^n.) are always near-perfect numbers.

The Fundamental Theorem of Arithmetic

Through their work in these investigations, students discover the Fundamental Theorem of Arithmetic. "Fundamental" theorems are few and far between in mathematics. The name implies that the theorem is of *fundamental* importance to mathematics as a field and, in this case, especially to number theory.

Fundamental Theorem of Arithmetic

Every positive whole number can be written as the product of primes in exactly one way, disregarding order.

Example

$120 = 2 \times 2 \times 2 \times 3 \times 5$

Although you can switch the order of the factors—i.e., you can write $120 = 2 \times 3 \times 2 \times 5 \times 2$—every prime product string for 120 will have three 2s, one 3, and one 5.

The Fundamental Theorem of Arithmetic helps us see why 1 is not a prime number. A whole number can be identified uniquely by its prime factorization. That is, each whole number corresponds to a unique prime factorization, and each prime factorization corresponds to a unique whole number. If 1 were a prime number, this would not be true. Any string of primes could be extended with an unlimited number of 1's.

Example

If 1 were a prime number, we could say that the "prime" factorization of 12 is

$3 \times 2 \times 2$ or
$3 \times 2 \times 2 \times 1$ or
$3 \times 2 \times 2 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$.

We could express 12 as a product using as factors as many 1's as we liked.

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Exponents

Exponents are introduced in this Unit as an efficient way to express repeated factors of a number.

> **Example**
>
> $12 = 3 \times 2 \times 2 = 3 \times 2^2$
> $2{,}700 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 = 2^2 \times 3^3 \times 5^2$

The **powers of 10** are $10^1, 10^2, 10^3, \ldots, 10^n$. In general, the powers of a number b are $b^n \cdot 10^3$ is said to be in **standard form** and $10 \times 10 \times 10$ is in **expanded form.** Students will later use exponents in *Filling and Wrapping* in the seventh grade and in *Growing, Growing, Growing* in the eighth grade.

Finding Prime Factorization

Another way to find the prime factorization of a number is to use a recording mechanism to help you keep track of the prime factors you have already found. Here is a recording scheme for the prime factorization of 100. Visit *Teacher Place* at mathdashboard.com/cmp3 to see the complete video.

Step 1 Find a prime factor of 100. Divide 100 by that factor, showing the work as an upside-down division problem.

Step 2 Find a prime factor of the quotient from Step 1. Divide the quotient by that factor, adding another "step" to the division problem.

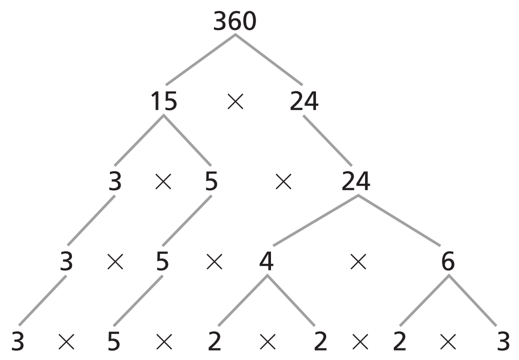
Step 3 Find a prime factor of the quotient from Step 2. Divide the quotient by that factor, adding a third "step" to the division problem.

Divide 100 by 2.	2	100
Divide 50 by 2.	2	50
Divide 25 by 5.	5	25
		5

The prime factorization of 100 is $2 \times 2 \times 5 \times 5$.

Another way to find the longest factorization of a number is to make a factor tree. This method is useful because it suggests "breaking apart" numbers into their factor pairs and subsequently "breaking apart" the factors. Note that some numbers can "break apart" into several possible factor pairs. For example, 360 equals 20×18 or 10×36. Below is one possible factor tree for 360.

Example



$$360 = 2^3 \times 3^2 \times 5$$

Students might develop a similar method on their own. If some students are having difficulty finding the longest prime factorization of a number, lead them to the factor tree method. The two methods are equally valid, so students should use the one that makes the most sense to them.

Common Factors and Common Multiples

Students are introduced to common factors and common multiples by exploring some interesting applications. In the beginning, they can find these by making lists of factors or multiples. The prime factorization of numbers is a more efficient method for finding common factors and common multiples.

In the *Applications—Connections—Extensions*, Venn diagrams are used to find common factors and multiples of two numbers.

Note: We do not ask students to fill in all of the numbers that fall outside the areas of the circles, but in a few cases we ask them to fill in some of those numbers.

The following illustrations are two Venn diagrams with all numbers less than or equal to 50 placed in their appropriate place. The Venn diagrams highlight the areas of overlap. The first diagram shows some of the common factors of the numbers. The second diagram shows some common multiples. This allows you to ask questions about what other numbers would be in the overlap if you kept checking numbers into the hundreds.

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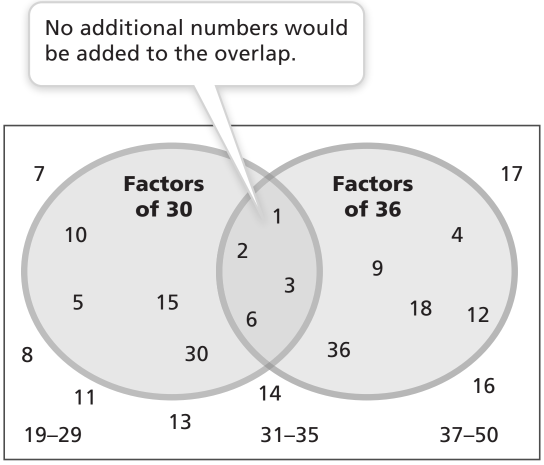
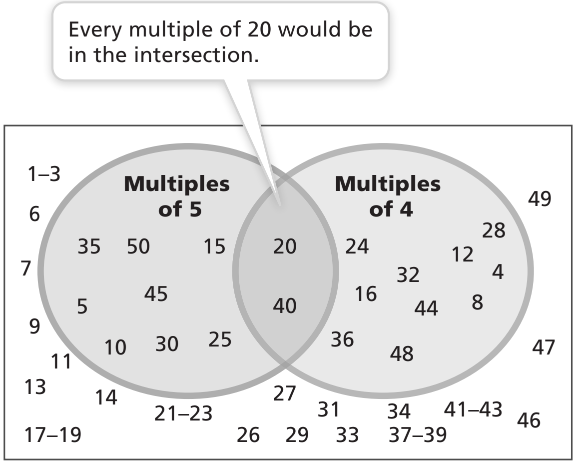




Using Prime Factorizations to Find the Greatest Common Factor and the Least Common Multiple

Once you have the prime factorizations of two numbers, you can use them to find the greatest common factor (GCF) and the least common multiple (LCM) of the two numbers.

Example

$72 = \underbrace{2 \times 2 \times 2 \times 3} \times 3 \qquad 120 = \underbrace{2 \times 2 \times 2 \times 3} \times 5$

Common prime factors

To find the GCF of 72 and 120, multiply the common prime factors to get $2 \times 2 \times 2 \times 3 = 24$. Thus, 24 divides evenly into 72 and 120, and no greater number does so.

To find the LCM of 72 and 120, find the union of all the prime factors. That is, multiply the common prime factors by each of the prime factors that 72 and 120 do not have in common. So the LCM of 72 and 120 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$.

Numbers with no common factors other than 1 are **relatively prime**. Examples are 72 and 35. The LCM of relatively prime numbers is the product of the numbers.

No common prime factors

$72 = 2 \times 2 \times 2 \times 3 \times 3$
$35 = 5 \times 7$

$\text{LCM} = 72 \times 35 = 2{,}520$

If you use the same strategy used earlier for 72 and 120, you multiply the prime factorization of 72 by the prime factorization of 35.

$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2{,}520$

The Relationship Between the GCF and the LCM

You can analyze the makeup of the LCM in another way. Look at the product of 72 and 120.

	72	×	120	= 8,640
Find the prime factorization of 72 and 120.	2 × 2 × 2 × 3 × 3	×	2 × 2 × 2 × 3 × 5	= 8,640
Circle the GCF and the LCM.	(2 × 2 × 2 × 3)	×	(3 × 2 × 2 × 2 × 3 × 5)	= 8,640
The GCF is 24 and the LCM is 360.	24	×	360	= 8,640
	GCF(72, 120)	×	LCM(72, 120)	= 8,640

We can use this example to find a rule for finding the LCM.

$\text{GCF}(72, 120) \times \text{LCM}(72, 120) = 72 \times 120$

$$\text{LCM}(72, 120) = \frac{72 \times 120}{\text{GCF}(72, 120)}$$

In general, $\text{LCM}(a, b) = \frac{a \times b}{\text{GCF}(a, b)}$. So, the LCM of two numbers *a* and *b* is the product of *a* and *b*, divided by the GCF of *a* and *b*.

This works because when you multiply two numbers that have common factors, there will be a factor string that repeats entirely within the product. This factor string is the GCF. If you divide this factor string out of the product, the result will have that factor string, occurring only once now, and any other non-common factors. This is the LCM.

Another feature of this Unit is the set of applied problems that engage students in using their knowledge of primes, factors, multiples, factor pairs, and square numbers. These problems create situations where the students figure out which of the things they have learned is appropriate to solve the problem. Selecting appropriate mathematical tools is critical for students to make use of what they know.

Classifying the Number 1

The question of how to classify and use the number 1 arises in many circumstances, including in prime factorization. Whole numbers can be written as a product of their prime factors. The Fundamental Theorem of Arithmetic states that this prime factorization is unique. In order to have a unique factorization for any whole number, we have to exclude 1 from the set of prime numbers. If 1 were defined as a prime number, there would be ambiguity about the prime factorization of a number; no factorization would be unique because we could make any string of factors longer by multiplying by 1.

> **Example**
>
> If 1 were prime, the number 14 could be written as any of the following:
>
> 7×2
> $7 \times 2 \times 1$
> $7 \times 2 \times 1 \times 1 \times 1$

Finding Greater Prime Numbers

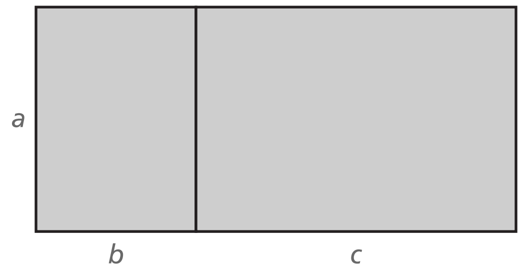
We know that the prime numbers grow sparser and sparser as whole numbers get greater, but is it possible that at some point prime numbers just "run out?" The answer is *no*. Visit *Teacher Place* at mathdashboard.com/cmp3 to see the complete video.

Investigation 1 has a *Did You Know?* about finding prime numbers with millions of digits. The book *The Mathematical Tourist* by Ivars Peterson also has a fascinating chapter called "Prime Pursuits" that might interest students.

Equivalent Expressions and the Distributive Property

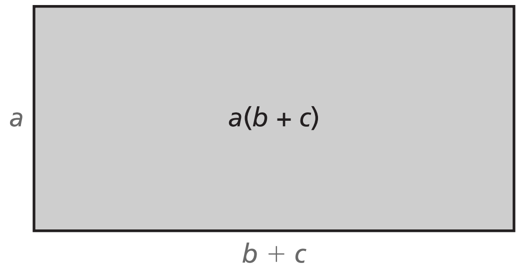
The Distributive Property is one of the most important and useful mathematical properties because it relates addition and multiplication. It states that a number can be expressed as both a product and a sum. Using the area of a rectangle as a visual representation helps students develop an understanding of the Distributive Property.

The numbers *a*, *b*, and *c* represent whole numbers. You can find the area of a rectangle in two different ways.



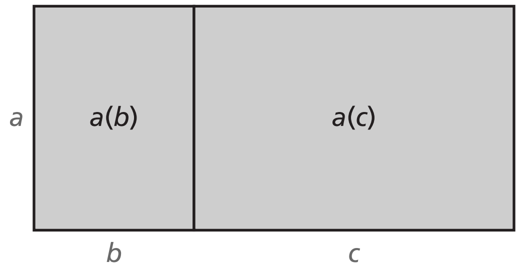
Method 1: Factored Form

Add the lengths of the two smaller rectangles to get the length of the large rectangle. Then multiply the length by the width to get the product $a(b + c)$. This expression is in **factored form**.



Method 2: Expanded Form

Find the area of each smaller rectangle. Then add the areas to get the sum $a(b) + a(c)$. This expression is in **expanded form**.



The two expressions, $a(b + c)$ and $a(b) + a(c)$, are **equivalent expressions** because the area of the large rectangle in both cases is the same. So, $a(b + c) = a(b) + a(c)$. This relationship is called the **Distributive Property**. Multiplication is distributed over addition. The Distributive Property also applies to subtraction.

Distributive Property

Algebra	Example
$a(b + c) = a(b) + a(c)$	$9(4 + 5) = 9(4) + 9(5)$
$a(b - c) = a(b) - a(c)$	$5(8 - 2) = 5(8) - 5(2)$

You can use the Distributive Property to understand the structure of multidigit multiplication.

Example

$$\begin{aligned} 7 \times 132 &= 7(100 + 30 + 2) \\ &= 7(100) + 7(30) + 7(2) \\ &= 700 + 210 + 14 \\ &= 924 \end{aligned}$$

This property also applies in geometric and contextual situations where both forms of the same expression describe the same quantity.

> **Example**
>
> The perimeter of a rectangle can be written as:
> $2\ell + 2w$ (doubling the length and adding double the width) or
> $2(\ell + w)$ (adding the length and width, then doubling)
> These expressions occur in *Covering and Surrounding*.

Students will later use the Distributive Property to develop algorithms for multiplication of rational numbers in *Accentuate the Negative* and to write equivalent algebraic expressions—as a product of two or more factors or as a sum of two or more terms—in *Variables and Patterns*.

Order of Operations

The order of operations is a mathematical convention to indicate which parts of an expression should be operated on in which order. This convention is essential so that expressions such as $4 + 6 \times 2$ are interpreted consistently and unambiguously.

Order of Operations
1. Work within parentheses.
2. Write numbers written with exponents in standard form.
3. Do all multiplication and division in order from left to right.
4. Do all addition and subtraction in order from left to right.

Parentheses have highest precedence because they allow for superseding the normal order of operations. For example, when evaluating the expression $4 + 6 \times 2$, multiplication is performed first, resulting in a value of 16. In the expression $(4 + 6) \times 2$, the parentheses require adding first, resulting in a value of 20.

Once students become comfortable with the Distributive Property, they may experience some confusion with the order of operations. In the following example, both methods are correct.

Method 1

Apply the order of operations.

$$\begin{aligned} 2(3 + 6) - 4 &= 2(9) - 4 \\ &= 18 - 4 \\ &= 14 \end{aligned}$$

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Method 2

Use the Distributive Property. Then apply the order of operations.

$$\begin{aligned} 2(3 + 6) - 4 &= 2(3) + 2(6) - 4 \\ &= 6 + 12 - 4 \\ &= 14 \end{aligned}$$

In the following example, it might be more efficient to write an equivalent expression first and then apply the order of operations.

$$\begin{aligned} 10 + 15\left(\frac{7}{5} + \frac{10}{3}\right) &= 10 + 15\left(\frac{7}{5}\right) + 15\left(\frac{10}{3}\right) \\ &= 10 + 21 + 50 \\ &= 81 \end{aligned}$$

An important idea to highlight is that you can apply the Distributive Property whenever an opportunity to distribute presents itself, and it is appropriate to do so. That is, you can use the Distributive Property to write an equivalent expression before applying the order of operations.

Example

The expression $2(n + 3) + 5$ cannot be simplified by using order of operations alone, but it can be simplified to $2n + 11$ by using the Distributive Property and addition.

$$\begin{aligned} 2(n + 3) + 5 &= 2(n) + 2(3) + 5 \\ &= 2n + 6 + 5 \\ &= 2n + 11 \end{aligned}$$

As students progress into simplifying algebraic expressions, this fluency with order of operations and the Distributive Property will continue to be useful.