Mathematics Background

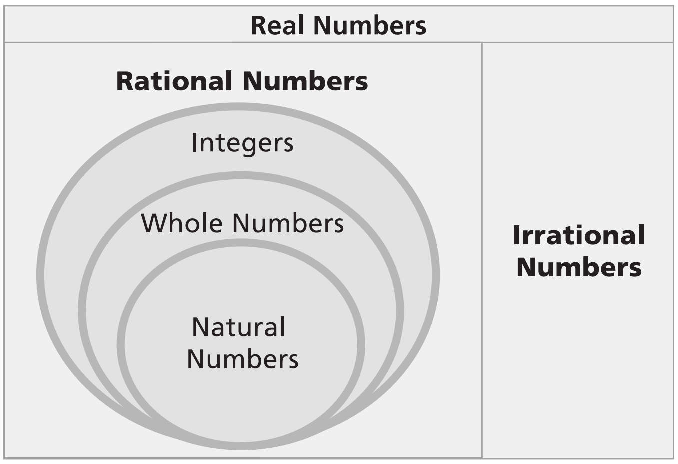
Rational Numbers

Students enter sixth grade with varying levels of understanding of fractions and equivalent fractions. In *Comparing Bits and Pieces*, students extend their understanding of fractions to the concepts of ratio and rate by exploring fractions as points on a number line, as measurements of distance, and in relative order. Students are introduced to rational numbers and ratio, a comparison of two quantities.

Careful sequencing of the problems in this unit helps students deepen their knowledge of fractions and feel more comfortable working with fractions. The problems in *Comparing Bits and Pieces* are the foundation for understanding ratio in later units and grades.

Rational numbers comprise a portion of the set of real numbers. Rational numbers are numbers that can be expressed as $\frac{a}{b}$, where a and b are both integers and b is not 0. From the point of view of sixth grade students this means that Rational numbers are positive and negative fractions, and any number that can be expressed as a positive or negative fraction. Thus, $4 = \frac{8}{2}$, and $0 = \frac{0}{5}$, and $-2\frac{1}{2} = -\frac{5}{2}$, are all rational numbers. This is all that sixth grade students need to know at this point.

Later they will encounter numbers that cannot be expressed as $\frac{a}{b}$, where a and b are integers and b is not 0. These numbers are the irrational numbers. The diagram shows that Irrational numbers and Rational numbers are mutually exclusive sets. Together the rationals and irrationals form the set of Real numbers.



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Rational numbers are described as a "dense" set, meaning that between any two rational numbers there is another rational number. For example, between $\frac{1}{100}$ and $\frac{2}{100}$, we can find another fraction, $\frac{15}{1000}$. Extending this idea we can see that there are actually an infinite number of fractions between $\frac{1}{100}$ and $\frac{2}{100}$, and between any two fractions. The rational numbers are countable, meaning that there is a one-to-one correspondence between rational numbers and natural numbers.

Proof that Rational Numbers are Countable:

To show that rational numbers are countable you have to show that you can put every one of them in a list, where each one has a unique and logical position in the list. In the sense that there is a unique rational in the first position on the list and another in the second position and another in the third position etc., you have made a countable list. (Countable means that you can make a one-to-one correspondence between each rational in the list and a counting number.)

In the diagram below, follow the arrows to see the order of the list created by Cantor. (Skip over fractions equivalent to prior fractions.) This accounts for every positive rational, and each rational only appears once in the list. To take care of the negative fractions you can insert each one behind its opposite in the list. This just leaves 0, and it can be given in the first position on the list.

	1	2	3	4	5	6	7	8	…
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	…
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	…
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	…
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	…
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	…
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	…
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	…
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

As a set, the rational numbers are the same "size" as the set {1, 2, 3, 4, . . .}. The irrational numbers are not countable—the set of irrational numbers is much "bigger" than the set of rational numbers. So much bigger, in fact, that the probability of drawing a rational number from a theoretical bag containing all the real numbers is 0. So it is surprising that a set as "small" as the rational numbers would be "dense" in the real numbers.

Inability to count the set of irrational numbers implies that its size is infinite. Another way to think about the relative sizes of the sets of rational and irrational numbers is this: Suppose you had a (theoretical) bag containing all the real numbers. The probability of drawing a rational number from the bag is 0. It is surprising, then, that a set as "small" as the rational numbers is "dense" on the real number line.

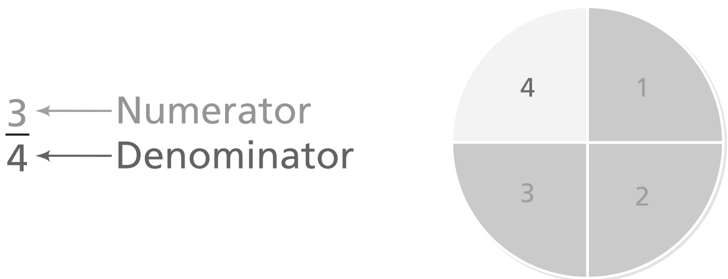
To learn more about characteristics of rational numbers, read *What Is Mathematics?* by Richard Courant and Herbert Robbins or *Journey Through Genius* by William Dunham.

As a way to talk about distance on the number line, the **absolute value** concept is introduced in Investigation 3. It is the value of a number when direction is not considered.

Check that students do not confuse the concept of absolute value with opposites. Students need to understand that absolute value measures distance from zero, which is always positive. Opposites are numbers that are the same distance from zero but in opposite directions on the number line.

Fraction Notation

The terms numerator and denominator are used in Investigation 1. Students examine *numerators* and *denominators* in contextual situations, so they need to understand the role of each.



The denominator refers to how many parts of equal size are in the whole. This is implicitly understood as relating to the size of the parts. For example, a fourths part is larger than a fifths part because the whole is partitioned into fewer parts, making the size of each part larger.

The numerator refers to how many of the parts are of interest or being referred to.

Look for opportunities throughout the unit to push students to use part-to-whole reasoning and to focus on the role of the numerator and the denominator. Help students understand the differences between situations in which it is useful to interpret the denominator as referring to the number of parts in the whole and those in which it is helpful to interpret the denominator as referring to the size of a part. This is a subtle distinction, but it is helpful in solving problems involving fractions. Also, it is important to students to be able to think about each of these roles when they explore fraction computation in *Let's Be Rational.*

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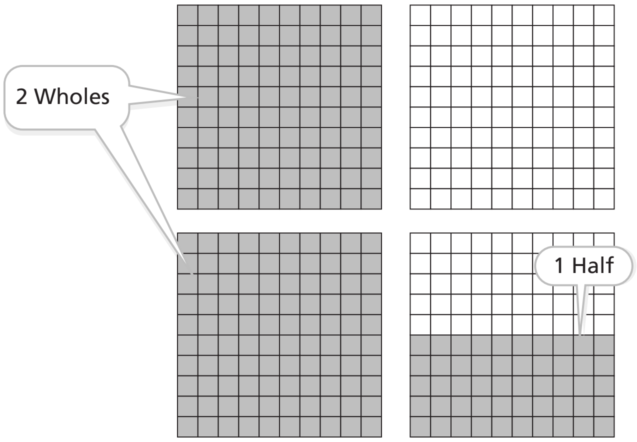
Numbers Greater Than or Equal to 1

Mixed Number

Mixed number is a useful term that applies to a number that is the sum of a whole number and a fraction. For example, $2\frac{1}{2}$ is a mixed number and means $2 + \frac{1}{2}$.

Model of $2\frac{1}{2}$

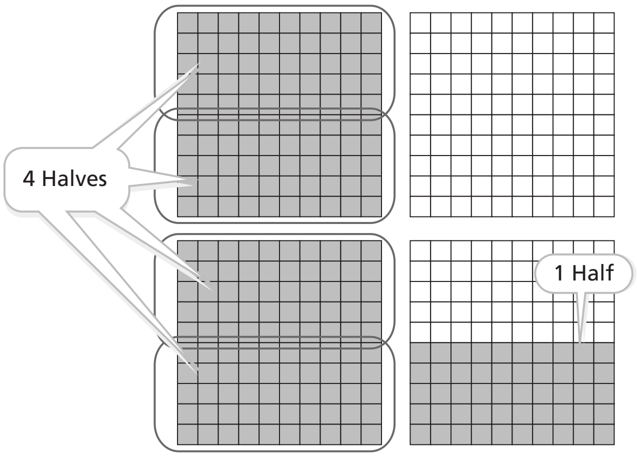


This is an important point to make with students so that they understand the conversions between mixed numbers and improper fractions.

Improper Fraction

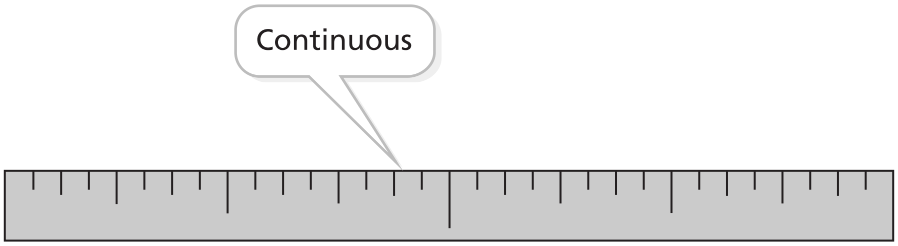
An **improper fraction** is defined as a fraction with a numerator equal to or greater than the denominator. In mathematics, the distinction between fractions and improper fractions is made simply for practical purposes and for helping students identify when they need to convert a mixed number to an improper fraction before multiplying or dividing. The important thing for students to understand is that when the numerator is equal to or greater than the denominator, the value of the fraction is 1 or greater than 1.

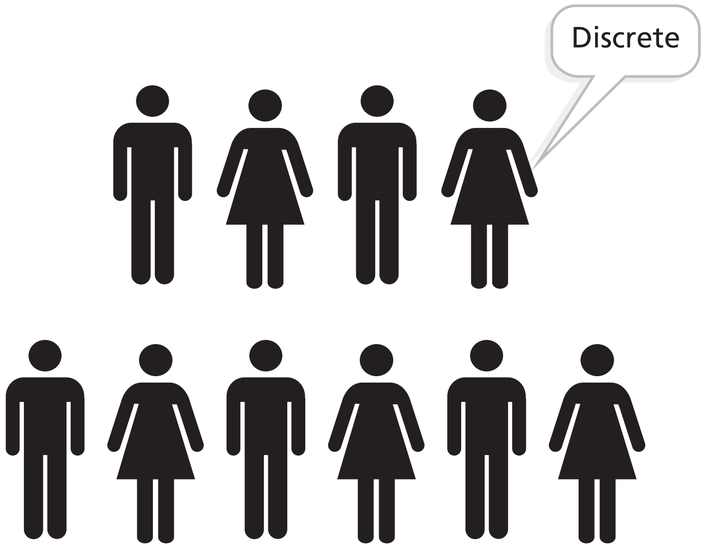
Model of $\frac{5}{2}$



Interpretations of Fractions

The interpretation of fractions is applied to both continuous and discrete numbers. Situations that are continuous are when measurements are taken, such as length or weight. Situations that are discrete are when counts of objects or people are recorded.





This unit focuses on interpreting fractions six ways.

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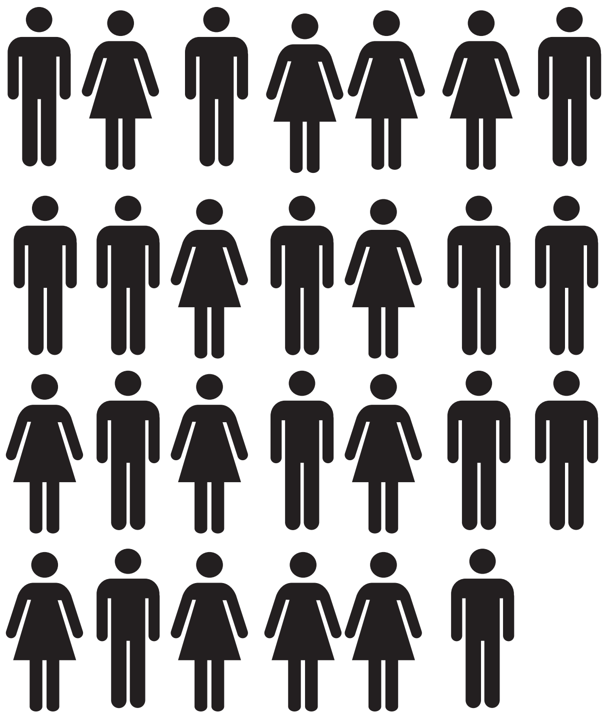
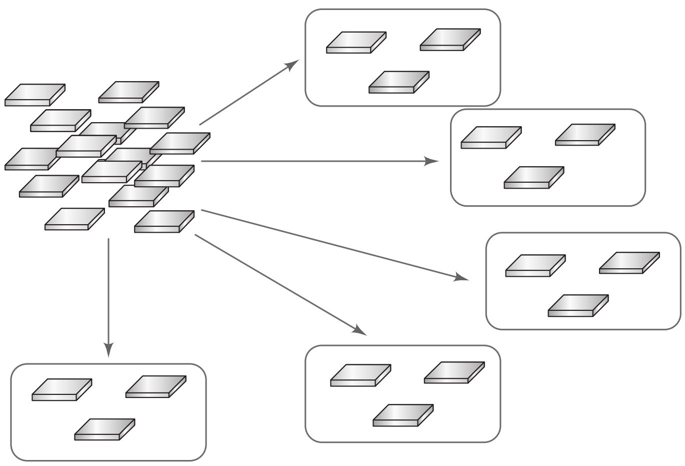
Parts of a Whole

Important to this interpretation is that it depends on partitioning an object or a set of objects into equal-size parts or groups for the purpose of making a comparison between the part and the whole set.

For example, if there are 27 students in the class and 13 are girls, the part of the whole that is girls can be represented as $\frac{13}{27}$.



Or, if you have 15 candies for 5 children to share, the fraction of the candies each child will get can be represented as $\frac{1}{5}$. The whole (15) is partitioned into 5 equal parts of 3. This can also be notated as $\frac{1}{5}$ of 15 = 3.



In the part–whole interpretation, students may have difficulty with one or more of the following.

- determining what the whole is
- subdividing the whole into equal-size parts—not necessarily equal shape, but equal size
- recognizing how many parts are needed to represent the situation
- forming the fraction by placing the parts needed in the numerator and the number of parts into which the whole has been divided in the denominator

Measures of Quantities

In this interpretation, a fraction is thought of as a continuous number commonly a measurement.

> Once there is an object to measure, the number representing the measurement can be "in between" two whole measures or somewhere along an interval. Students see this every day in references such as 11.5 million people, or $\frac{5}{8}$ of an inch. This interpretation is important for students' mathematical development because it leads to the comparing and ordering of fractions, and correctly performing operations on fractions.

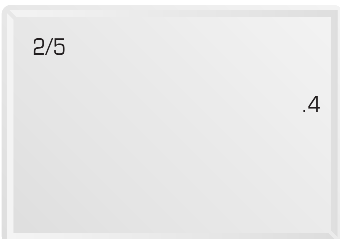
Indicated Divisions

To move easily between fraction and decimal representations of rational numbers, students need to understand that fractions can be thought of as indicated divisions. To help students see how this interpretation is related to whole-number division, show that sharing 12 apples among 3 people calls for division and so does sharing 3 apples among 4 people.

12 apples ÷ 3 people = 4 apples each   3 apples ÷ 4 people = $\frac{3}{4}$ of an apple each

Decimals

A byproduct of the division interpretation of fractions is the relationship between a fraction and its decimal representation. The decimal representation of the fraction $\frac{2}{5}$ is found by dividing 2 by 5.



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Calculators and computers provide decimal representations quickly, making decimals more common today than in the past. Students need time and experience moving between fractions and decimals in order to feel comfortable working with either. They also need to understand decimals in two ways:

- as special fractions with denominators that are powers of 10
- as an extension of the place-value system for representing quantities less than 1

When children study whole-number place value, they learn to decompose numbers according to place value.

$25 = 2(10) + 5(1)$

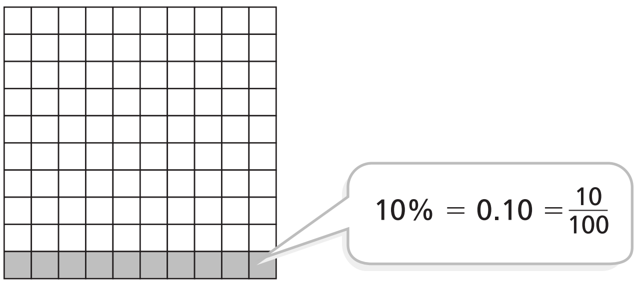
For fractional (decimal) place value, students need to be able to decompose decimal numbers into fractions.

$0.25 = 2(0.1) + 5(0.01)$

This reasoning helps students understand the situations in which *tacking a zero onto the end of a decimal does not change its value*. Writing 0.25 as 0.250 is not "adding a zero." Instead, it is rewriting the decimal in an equivalent form, as 2 tenths plus 5 hundredths plus 0 thousandths.

Percents

Rather than treating fractions, decimals, and percents as separate topics, this unit seeks to build connections among them. Students will see that the ideas and concepts are related and the differences are in the symbols used to represent those ideas. **Percents** are introduced as special names for parts of 100.



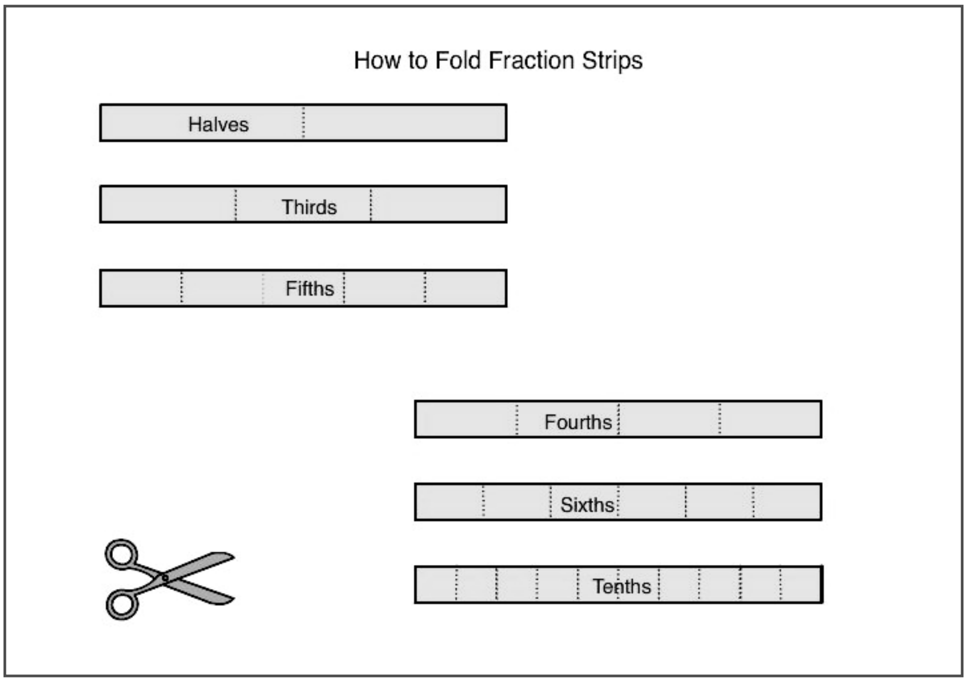
These interpretations of fractions, together with summaries of research on how children interpret and think about fractions, are summarized in the book *Extending Children's Mathematics* by Susan Empson and Linda Levi.

Models of Fractions, Decimals, and Percents

Throughout this unit, rational numbers are modeled various ways.

Fraction Strips

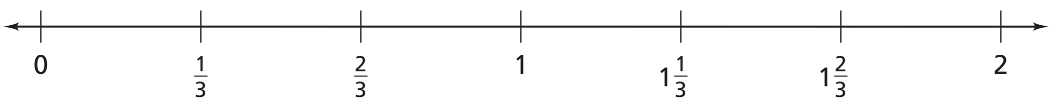
In Problem 1.3 students are introduced to fraction-strip models. Fraction strips can be created by dividing or folding a strip of paper into equal-size parts. Visit Teacher Place at mathdashboard.com/cmp3 to to see the complete video.



Number Lines

To motivate a number-line model, fraction strips are labeled at the divisions or folds. The number-line model helps connect fractions to quantities and measurements. It also supports students' understanding of partitioning into and naming smaller and smaller parts.

This is a number line from 0 to 2 with a few fraction quantities marked:



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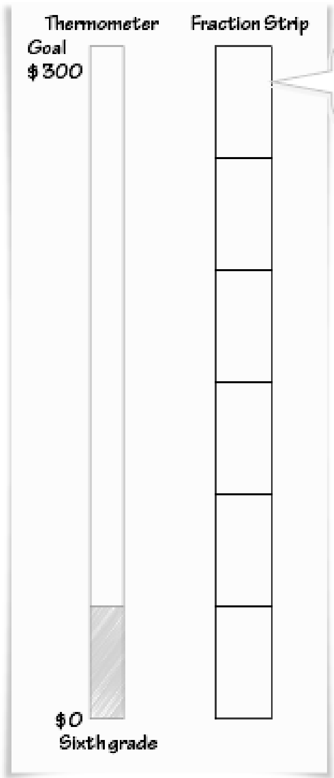
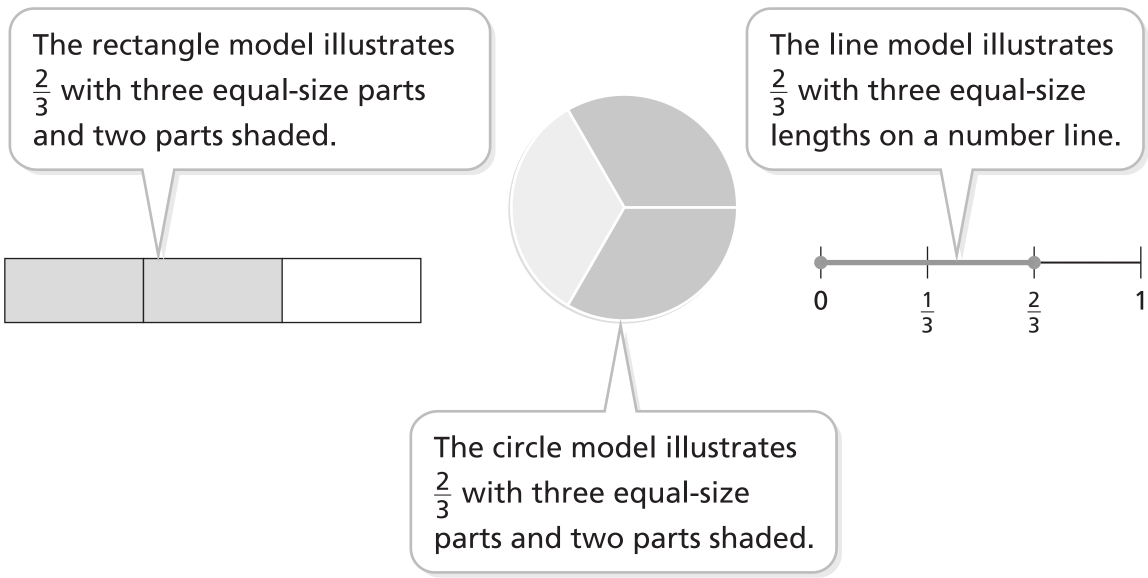
Partitions

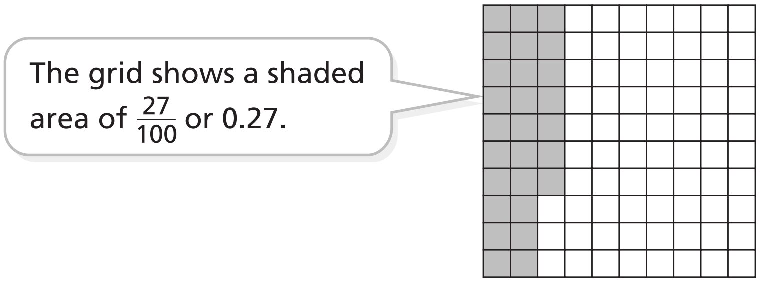
A more general model of fraction situations based on partitioning an area is used in Investigation 1.





Grid Areas

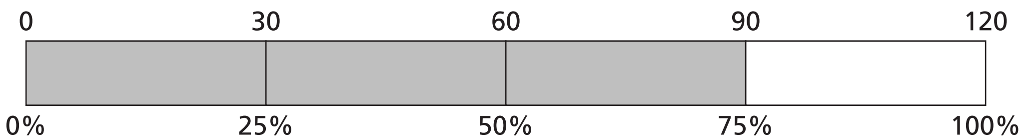
Because 100 and powers of 10 are useful in understanding fraction and decimal relationships, grid-area models are introduced in Problem 3.3 and further developed throughout the unit.



Percent Bars

A percent bar partitions a rectangle into equal-size parts and then labels the top side by quantity and the bottom side by percent. The two scales on the top and the bottom are related—they show the parts of the total that are equivalent to various percents.

90 is equivalent to 75% of 120

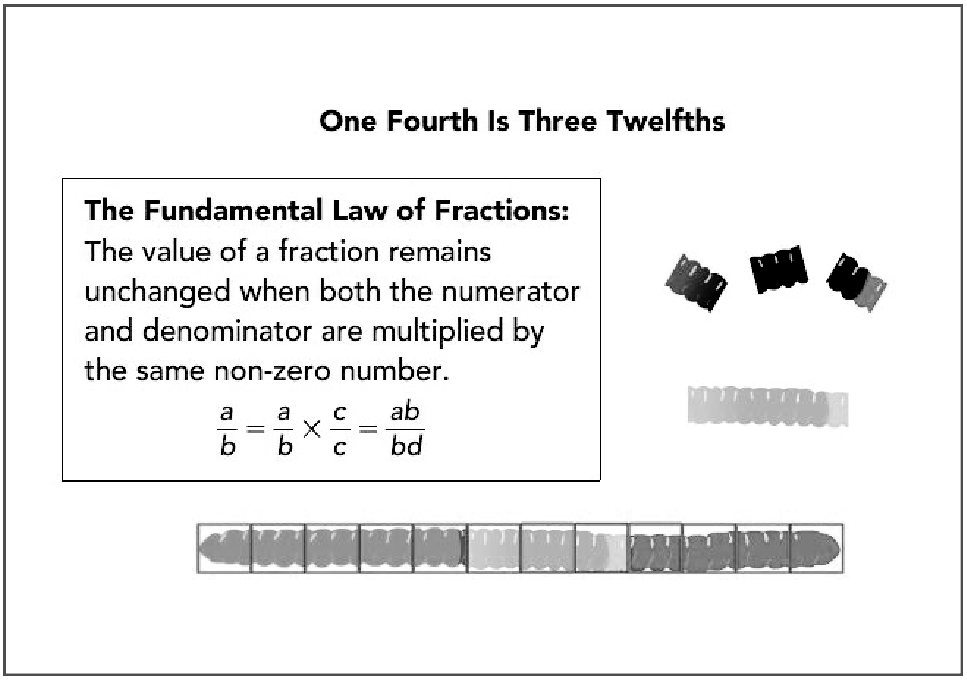


These models were chosen because they connect directly to the interpretations of rational numbers in the unit raises.

Note: The Common Core State Standards uses the term *tape diagrams* to encompass linear models. The tape diagram is also known as a strip diagram, bar model, fraction strip, or length model. In *Comparing Bits and Pieces*, students use a variety of linear/length models to express fraction, percent, and decimal relationships. The term *tape diagram*, a non-standard term in American curriculum, appears in Problem 1.5.

Equivalence of Fractions—Fundamental Law of Fractions

Partitioning and then partitioning again is an important skill that contributes to understanding **equivalent fractions**. This property of equivalence is used to add and subtract fractions with different denominators. Visit Teacher Place at mathdashboard.com/cmp3 to to see the complete video.



For example, if a worm is marked into fourths (the first partition) and then each fourth is marked into thirds (the second partition), each original fourth has three parts (or three twelfths) in it. Thus one fourth is equivalent to three twelfths. Because each fourth was marked into three parts, the size of the parts (denominator) is one third of the original fourth (which is $\frac{1}{12}$) and the number in the whole (denominator) tripled. It will take three times as many of the new pieces (numerator) to create the original one-fourth.

That one multiplies by three when changing from fourths to twelfths numerically is because of repartitioning the original pieces. When we start with fourths and partition each fourth into three parts, the smaller subdivisions are twelfths, because the whole is now divided into twelve parts; moreover, for each one fourth there are now three twelfths. That is both the numerator and the denominator of $\frac{1}{4}$ are multiplied by 3 in the renaming caused by repartitioning. As students look for patterns in their partitioning, help them to see the relationship developed from repartitioning. In *Let's Be Rational*, students will apply this idea as the algorithm of equivalent fractions is formally developed.

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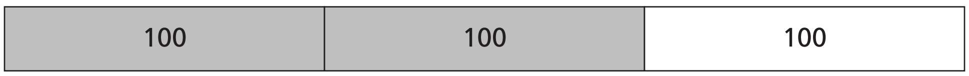
Equivalence of Ratios

Equivalence of ratios is a lot like equivalence of fractions. If two fractions are equivalent, they are always equivalent. They remain equivalent no matter whether we interpret them as part–whole relationships, as numbers, or as ratios.

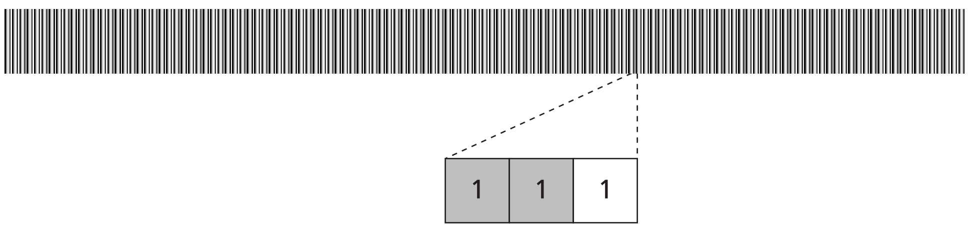
Ratios are **comparison statements**. They can be rewritten as equivalent ratios, but as shown below there are two ways to interpret the meaning of their equivalence.

Suppose the sixth-graders raised \$200 toward their \$300 fundraising goal. Then we can say that the ratio of the money they have raised to the money they have left to raise is 200 : 100. Using equivalent fractions, we can also say that this ratio is 2 : 1.



One way to think about this is that the 2 represents two groups of \$100 and the 1 represents one group of \$100. In this interpretation, the 2 : 1 compare parts of the whole.

Another way to think about the equivalence of the ratios 200 : 100 and 2 : 1 is for every \$2 the sixth-graders have raised, they still have to raise \$1. In this interpretation, the numbers in the ratio count dollars. This is shown in the bar model below: each shaded part represents \$2, each unshaded part represents \$1.



Fraction Benchmarks

Benchmarks are numbers that are easy referents such as 0, $\frac{1}{2}$, and 1. One way to estimate the size of a fraction is to compare it to these benchmarks. The set of benchmarks is expanded as students work through the unit to include $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{10}$ in fraction, decimal, and percent form. The goal of this work on benchmarks is to develop a strategy for estimating relative sizes of numbers whether in fraction, decimal, or percent form. In developing operations in Let's Be Rational and *Decimal Operations*, these ideas are the key to estimating sums and products. Students need to know the benchmark list in all forms to become good estimators.

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Fractions Between Fractions

Being distributed on a number line so that between any two fractions there is another fraction makes fractions quite useful in measurement contexts because the partitioning allows finer and finer grained measures or we say that rational numbers are "dense" in the set of real numbers. See **Rational Numbers**.

Place-Value Notation for Decimal Fractions

Fraction notation makes clear to the reader the number of units and the size of the unit being used. For example, the denominator of the fraction $\frac{4}{5}$ indicates that the size of the unit, fifths, and the numerator indicates that four fifths are being used or referenced. The same can be said of the fraction $\frac{8}{10}$.

Fundamental Law of Fractions

$$\frac{4}{5} \times \frac{2}{2} = \frac{4 \times 2}{5 \times 2} = \frac{8}{10}$$

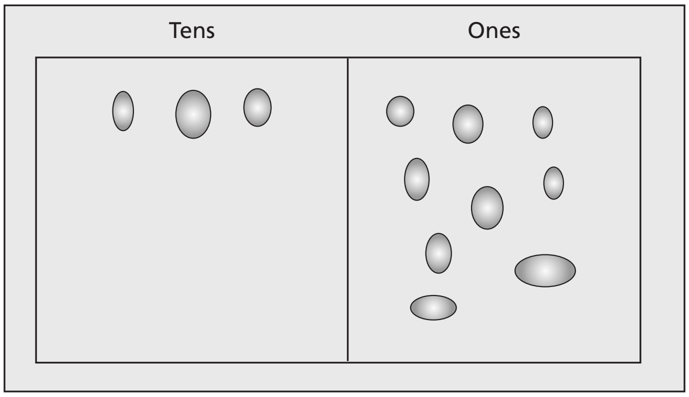
By simply reading the fraction, i.e., "eight tenths," we know that we are working with tenths, and that we have 8 of them. In base-ten notation, this is less obvious due to the language used to read decimal numbers, i.e., "point eight."

When using base-ten notation to write numbers, the size of the unit is implied by the place of the digit or its relative position to the decimal point. For example, 0.8 is read "eight tenths" and represents eight tenths because the digit 8 is in the first place to the right of the decimal. That place, or position, is always designated for tenths. The digit in this position represents the number of fractional parts of one whole when the whole is partitioned into 10 parts.

The base-ten system for writing numbers was adopted from the Hindu-Arabic numeral system. This system was used in Persian after Al-Khwarizmi's (c. 825) book *On the Calculation with Hindu Numerals* and used by Arab mathematicians after Al-Kindi's (c. 830) volumes *On the Use of the Indian Numerals*. Fibonacci brought the Hindu-Arabic numeral system to western world c. 1200, which was one of the major reasons for the expansion of trade and mathematics prior to the Renaissance.

The base-ten system is a compressed method of writing the expanded form of a number.

$3(10) + 9(1) = 39$



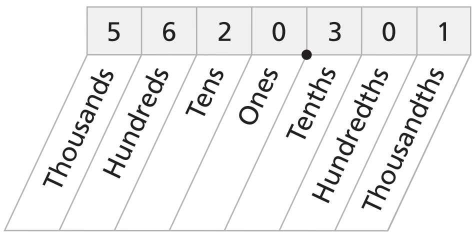
The greater the number, however, the clumsier and more inefficient both the abacus and expanded form become.

$$2(10{,}000) + 8(1{,}000) + 5(100) + 9(10) + 0(1) = 28{,}590$$

In the compressed base-ten form, all mathematical operations are still possible with a few algorithms, which students have studied in prior grades.

The Hindu-Arabic mathematicians realized they needed to extend the number system to represent numbers less than 1. At first, they simply divided an abacus into two parts, with one section to represent whole numbers on the left and the other section on the right to represent fractions.

Later, a decimal point was used to separate the digits so that the numbers to the right of the decimal point represented fractions with denominators of 10 (tenths), 100 (hundredths), 1,000 (thousandths), 10,000 (ten-thousandths), and so on.



Ordering Decimals

One advantage of decimals over fractions is that they are easy to order. In general, to find the greater of two decimal numbers, compare them one digit at a time. If the whole number portion is equal, continue to compare the fractional part one digit at a time. Start with the greatest decimal place, i.e., the first place to the right of the decimal point, and work one digit at a time to the right. Look for the first place where the two numbers differ. The number with the greater digit in that place is greater.

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Illustrate with 2,002.13 is larger than 2,002.016 because 2,002.13 has a 1 in the tenths place, while 2,002.016 has a 0 in the same place.

In the tenths place, 1 is greater than 0.

2,002.13 is greater than 2,002.016

Ordering decimals does not require the same kind of creative and flexible thinking that ordering fractions does. In prior grades, students may have come to understand that a longer whole number is greater. Logically they may want to use this rule for comparing decimal numbers, too. With decimals, this may not be true, and students will need to learn to align the decimal points and compare digits with the same place value before deciding which number is greater.

Ratios

A ratio is a comparison of two quantities that depends on multiplication and division. A *difference*, by contrast, is a comparison of two quantities that depends on addition and subtraction. Sometimes it makes sense to compare with differences, and sometimes it is better to compare with ratios.

Mathematically, students typically come to sixth grade thinking additively, and they will tend to compare using differences. An important goal of the proportional reasoning work in middle school is to improve students' multiplicative thinking; this includes developing skills in using ratios to compare.

In Investigation 1, the context of fund-raising by sixth, seventh, and eighth grades is used to introduce "for every" statements to make ratio comparisons.

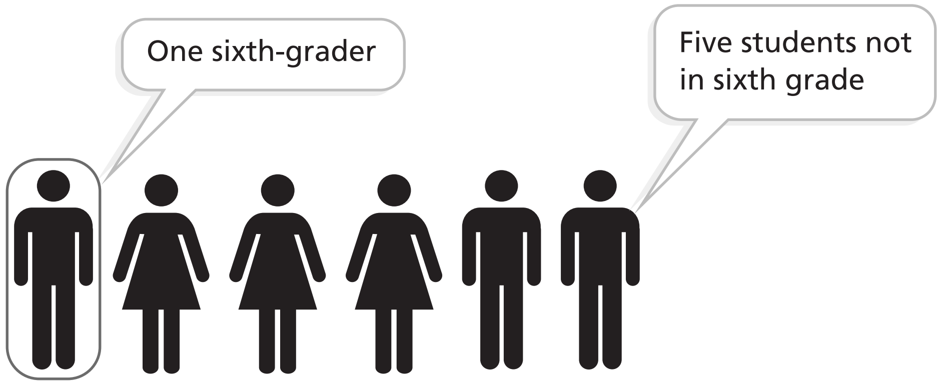
<table>
<tr><th colspan="2">For every $60 the sixth-graders plan to raise, the seventh-graders plan to raise $90. We can rewrite this comparison as:</th></tr>
<tr><td colspan="2">The ratio of the sixth-grade goal to the seventh-grade goal is 60 to 90.</td></tr>
<tr><th colspan="2">We can also write ratios, with a colon (:)</th></tr>
<tr><td>The ratio of the sixth-grade goal to the seventh-grade goal is 60 to 90.</td><td>The ratio of the sixth-grade goal to the seventh-grade goal is 60 : 90.</td></tr>
</table>

We can also use "per" to express ratios.

Example

Mary runs at 5 miles per hour.

Ratios are related to fractions as operators ("stretchers" or "shrinkers") and as rates or parts of a proportion. An example is if a school principal states that $\frac{1}{6}$ of the school is sixth-graders.



Strictly speaking, this is a comparison of two numbers or two measurements, a ratio, and not a single measurement. The comparison is a part to the whole, i.e., one out of every six students is a sixth-grader. Without knowing how many students are in the school, though, we do not know if there are more sixth-graders in this school than another school where $\frac{1}{5}$ are sixth-graders. This is a situation in which percents are used to express ratios.

Unit Rates

A unit rate is a comparison of two quantities in which the result is expressed as a ratio like "3 of A for every 1 of B." Unit rates are often expressed using decimals. For example, "fuel efficiency of 24.6 miles per (one) gallon of gasoline" or "food cost of $2.89 per (one) pound."

Example

Mary runs 10 miles in 2 hours
Mary runs 2.5 mile in a half hour (or 30 minutes.)
Unit Rate: Mary runs 1 mile in 1/5 hour (or 12 minutes.)

Rate Tables

Another way to organize equivalent ratios is in rate tables. **Rate tables** are similar to multiplication tables. They are extended from a pair of rows (or columns) from the multiplication table.

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For example, if 1 ounce of popcorn kernels yields 4 cups of popcorn, a rate table can be used to calculate equivalent amounts of popcorn from other amounts of kernels.

Popcorn Table

Cups of Popcorn	4	8	12	16	20	24	28	32	36	40	44	48
Unpopped Popcorn Kernels	1	2	3	4	5	6	7	8	9	10	11	12

The student council also decides to sell popcorn to raise money. One ounce of popcorn kernels yields 4 cups of popcorn. One serving of popcorn will be a bag with 2 cups of popcorn. Use a rate table to find the ounces of popcorn kernels needed for the number of bags of popcorn.

Popcorn Table

Bags of Popcorn	1	2	4	6	8	10	12	14	16	18	20	22	24
Cups of Popcorn	2	4	8	12	16	20	24	28	32	36	40	44	48
Unpopped Popcorn Kernels	$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	11	12

The following questions can be answered by using a rate table and scaling up or multiplying by a common factor.

How many cups of popcorn can be made from 12 ounces of popcorn kernels? From 30 ounces?

How many ounces of popcorn kernels are needed to make 40 cups of popcorn? 100 cups?

How many ounces of popcorn kernels are needed to make one cup of popcorn? (**Note:** The answer is a unit rate.)

We can write ratios as fractions, e.g., to answer the last question, $\frac{1 \text{ bag}}{\frac{1}{2} \text{ oz}} = \frac{2 \text{ bags}}{1 \text{ oz}}$.

From the table, 4 cups of popcorn kernels makes 16 cups of popcorn. This can be written as $\frac{4}{16}$ or $\frac{1}{4}$ using equivalence of ratios. Doing this shifts the interpretation from a part–whole fraction to a comparison of two measurements.

For most of this unit, ratios are not written as fractions intentionally. The notation for part–whole fractions and rational numbers is kept separate from the notation for ratio comparisons. This is intended to help teachers and students separate these ideas as they are developed, and to reduce confusion.

Usually when we use the word *fraction*, we mean the *part–whole* or *number* interpretation of the fraction. Fraction notation can be used to represent many different ideas. This becomes most challenging when interpreting fractions as ratios. In seventh grade, fraction notation will be used to write ratios in *Stretching and Shrinking* and *Comparing and Scaling*.