Mathematics Background

The Measurement Process

While this Unit does not focus on the global aspects of what it means to measure, it does raise issues that help students see relationships and characteristics of all measurements.The measurement process involves several key elements.

- A phenomenon or object is chosen, and an attribute that can be measured is identified.

> **Example**
>
> Measurements typically involve such properties as height, mass, time, temperature, and capacity.

- An appropriate unit of measure is selected. The unit depends on the kind of measurement to be made and the degree of precision needed for the measurement.

> **Example**
>
> Units of measure include centimeters, angstroms, degrees, minutes, volts, and decibels. Instruments for measuring include rulers, calipers, scales, watches, ammeters, springs, and weights.

- The unit is used repeatedly to "match" the attribute of the phenomenon or object in an appropriate way.

> **Example**
>
> This matching might be accomplished, for example, by "covering," "reaching the end of," "surrounding," or "filling" the object.

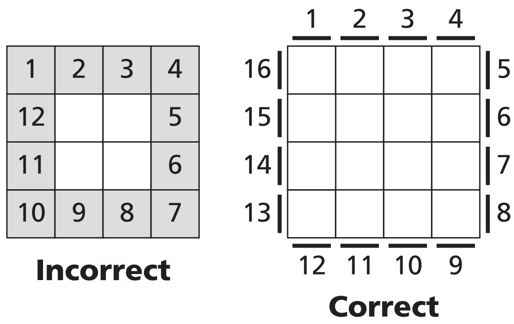
- The number of units is determined. The number of units is the measure of the attribute of the phenomenon or object.

Measuring Perimeter and Area

Covering and Surrounding highlights two important kinds of measures, perimeter and area, that depend on different units and measurement processes. Counting is a natural and appropriate way for students to find area and perimeter, because measurement is counting. When you measure, you are counting the number of measurement units needed to "match" an attribute of an object.

Measuring perimeter requires linear units. Measuring area requires square units. When finding the perimeter of a figure, students will often say they counted the number of squares along a side to find the length. Students need to be aware that perimeter is a linear measure. To measure the perimeter, you count (measure) the number of unit lengths that form the border of the figure.

In the figure at the left below, the 12 square tiles border a 4-by-4 square. This is not perimeter. Instead, the perimeter comprises 16 unit lengths, shown in the figure at the right below.



If there is a strong emphasis on formulas preceding understanding, then the methods may contribute to this confusion of perimeter and area. While students can become adept at plugging numbers into formulas, they often have a hard time remembering which formula does what. This is often because they have an incomplete fundamental understanding of what a measurement is and how a formula captures their more informal, intuitive computations.

Many students think that area and perimeter are related in that one determines the other. They may think that all rectangles of a given area have the same perimeter or that all rectangles of a given perimeter have the same area. Alternatively, students may not see any distinction between area and perimeter, giving area answers for perimeter problems or vice versa. The Investigations in *Covering and Surrounding* help students realize for themselves the inaccuracy of such notions and help them to understand the distinctions between the two measures.

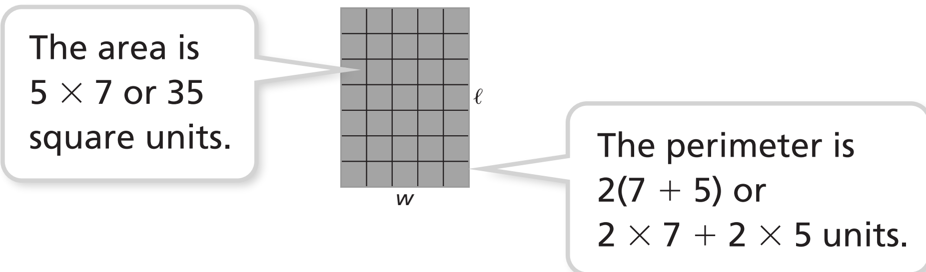
In this Unit, students work with tiles, grid paper, string, rulers, and other devices of their choice to develop a dynamic sense of *covering* and *surrounding* as ways of finding area and perimeter. Once students have an understanding of area and perimeter, they are ready to develop rules or formulas for finding area and perimeter in certain situations. This should be encouraged, but not forced too early. Some students need the help of a more hands-on approach to measuring for quite a while. The payoff for allowing students the time and opportunity to develop levels of abstraction with which they are comfortable is that they will eventually make sense of perimeter and area in a lasting way.

Area and Perimeter of Rectangles

In Investigation 1, students first explore area and perimeter of nonrectangular shapes. After building shapes with square tiles and computing the perimeter and area by counting the units, the students investigate rectangles displayed on a grid. Again they find that they can measure the area by counting the number of squares enclosed by the rectangle and the perimeter by counting the number of linear units surrounding the rectangle.

Students may have found that, once you have counted the grid squares in one row, you can multiply by the number of rows to find the total number of squares in the rectangle. In other words, you can find the area of a rectangle by multiplying the length by the width.

For example, in this rectangle there are 5 squares in the first row and 7 rows in all.



In general, the formula for the area of a rectangle is $A = \ell \times w$. Similarly, the formula for the perimeter of a rectangle is $P = 2(\ell + w)$ or $P = 2\ell + 2w$.

Parentheses and Order of Operations

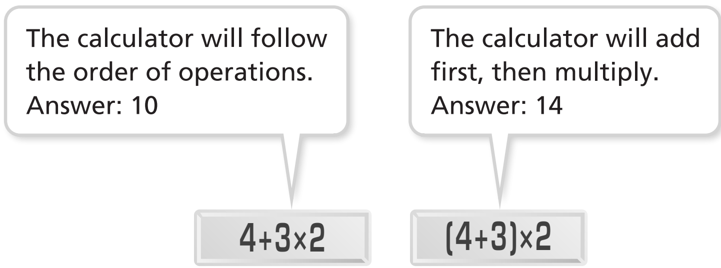
The two equivalent formulas for the perimeter of a rectangle provide an opportunity to discuss both the role of parentheses in expressions and the order of operations. Parentheses indicate that the numbers within them need to be operated on first.

perimeter = (length + width) × 2

Add the length and width before multiplying by 2.

Students who are using a scientific calculator, or one that follows order of operations, need to be aware that they must enter the string correctly for it to give the correct answer.

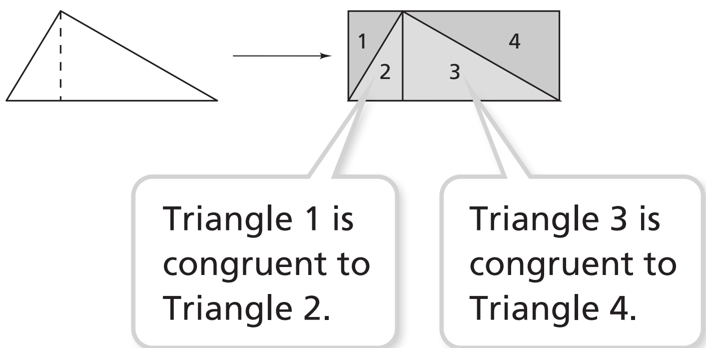
For example, for a rectangle with a length of 4 and a width of 3, if you enter the string of numbers and operations as $4 + 3 \times 2$, the calculator will automatically follow order of operations and multiply 3×2 before it adds the 4. If you want to add $4 + 3$ first, either key in $4 + 3$ and press equal to get the sum before keying in to multiply by 2, or use the parentheses keys, as shown in the bottom right.



Area of Triangles

In Investigation 2, students use their knowledge about finding area and perimeter of a rectangle to find the area and perimeter of a triangle. Any triangle can be thought of as half of a rectangle.

If you surround a triangle with a rectangle in a particular way, there are two small triangles that are inside the rectangle and also outside the triangle (triangles 1 and 4 in the diagram). Point out to students that when you draw in the height of the triangle, the two triangles created (triangles 2 and 3 in the diagram) are congruent to the other two triangles in the rectangle.



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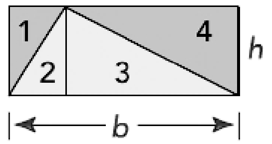

area of $\triangle 1$ + area of $\triangle 2$ + area of $\triangle 3$ + area of $\triangle 4$ = area of a rectangle

2(area of $\triangle 2$) + 2(area of $\triangle 3$) = area of a rectangle

area of $\triangle 2$ + area of $\triangle 3$ = $\frac{1}{2}$ · area of a rectangle

area of original triangle = $\frac{1}{2}$ · area of a rectangle

The area of the original triangle is $\frac{1}{2}b \times h$, where b is the base of the triangle (or the length of the rectangle) and h is the height of the triangle (or the width of the rectangle).

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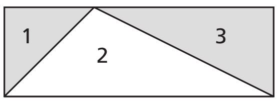
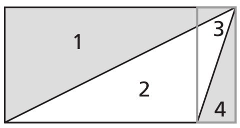
Hence the area of the original triangle is $\frac{1}{2}b \times h$, where b is the base of the triangle (or the length of the corresponding rectangle) and h is the height of the triangle (or the width of the rectangle).

Obtuse triangles are not directly addressed in this Unit because every obtuse triangle has one orientation where the smallest upright rectangle does not have an area equal to twice the triangle. In the first arrangement below, the rectangle and the triangle have the same base and height. In the second triangle, the base of the obtuse triangle is shorter than the base of the enclosing rectangle.

Same Base



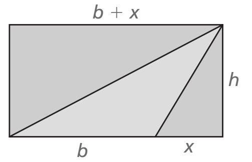
Shorter Base



The area of the small rectangle is *not* twice the area of Triangle 3.

The formula for area of a triangle still holds for obtuse triangles, regardless of orientation. However, the approach modeled in Problem 2.1 does not demonstrate why this is so. Here is one proof of why the formula works for obtuse triangles.

The following obtuse triangle has a base b and a height h. It is embedded in a rectangle. Note that the bottom side of the rectangle is made up of two parts, the base of the obtuse triangle, which is b, and the base of a right triangle, which is x. From the preceding demonstrations, the area of a right triangle is known. So the length or base of the rectangle is $b + x$ and its height is h.



area of the obtuse triangle = area of the rectangle − area of the two other triangles

$$= h(b + x) - \frac{h(b + x)}{2} - \frac{hx}{2}$$

$$= \frac{2h(b + x) - h(b + x) - hx}{2}$$

$$= \frac{2hb + 2hx - hb - hx - hx}{2}$$

$$= \frac{hb + 2hx - 2hx}{2}$$

$$= \frac{hb}{2}$$

Area of Parallelograms

The rule for area of a parallelogram is developed from students' understanding of the rule for finding area of triangles. This may be a different approach than the one you have used to develop a rule for area of a parallelogram. It is not uncommon to see the rule for area of a parallelogram developed out of the rule for area of a rectangle.

When students informally explore ways to find the area of parallelograms, they often report that they can cut the parallelogram and rearrange it into a rectangle. They can then use their rule for area of a rectangle to find the area of a parallelogram. This approach can be illustrated with this diagram:

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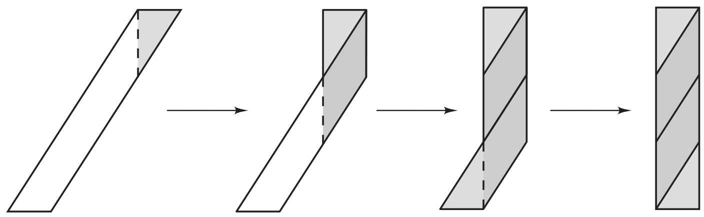
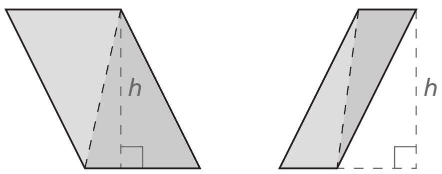
While this method works for most parallelograms, and it should be accepted when students offer it as an informal strategy, it will not work with parallelograms like the one shown. It is not possible to make one vertical cut and rearrange the pieces to form a rectangle. (Of course, one can reorient the parallelogram so that the longer side is the base. Then the rearrangement works.)



However, every parallelogram can be divided into two congruent triangles by drawing one diagonal on the parallelogram, as shown. When the parallelogram is divided this way, the parallelogram has the same base length and height as the two triangles. Examples of two parallelograms are shown.



The height of a parallelogram is the perpendicular distance from the base to the side parallel to the base. As is the case with triangles, the height of a parallelogram depends on the side that is chosen for the base.

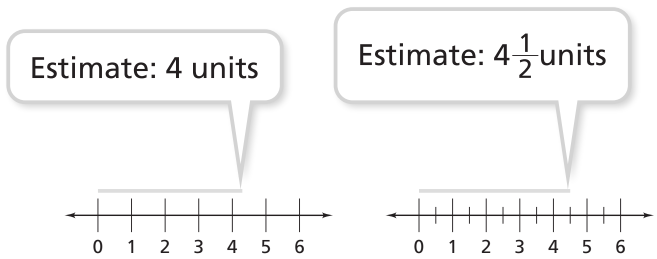
From this process, students can see that they find the area of the parallelogram by multiplying the base and height, without dividing by two, as they did when finding the area of a triangle. Thus, the area of a parallelogram is $2 \times \left(\frac{1}{2}b \times h\right)$, or just $b \times h$.

Estimating Perimeter and Area of Irregular Figures

Finding an exact measurement for area and perimeter is not always possible or even necessary. However, there are methods that allow for more precise estimates. In the ACE of Investigation 1, there are a few exercises that explore finding the area of irregular shapes using the context of lakes and shorelines in order to explore ways to find reasonable estimates for perimeter and area. The amount of precision in the estimation is the subject of advanced mathematics courses.

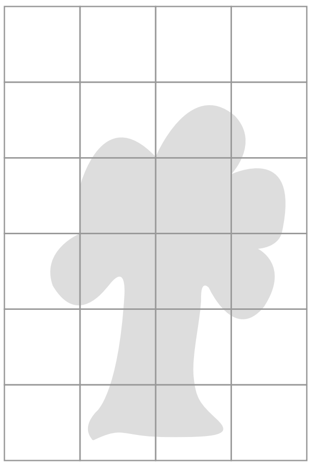
When estimating perimeter, students use a string to wrap around the picture of the lake and then measure the length of the string with the appropriate linear measure. When measuring length, there is always some estimation. Once the unit of length has been selected, it can be subdivided to make more accurate measurements because there is less room for estimation error.

For example, suppose you estimate the length of the segment measured with two scales below. With the second measurement scale, you can make a more precise estimate.



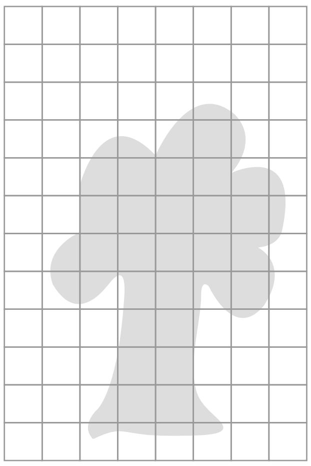
To measure the surface area of the lake, students select a corresponding square unit with which to cover the surface. The number of units needed to cover the lake is the area. Just as with length, using smaller units for measuring area results in less estimation error and gives more accurate answers. In the following pictures, the object is measured first with centimeter squares and then with half-centimeter squares.

Centimeter Grid



Estimate: $7 cm^2$

Half-Centimeter Grid



Estimate: $7\frac{1}{2}$ cm^2

From the pictures, you can see more of the tree is covered with complete half-centimeter squares, leaving less room for error around the edges.

Accuracy and Error

Measuring objects and comparing data from the whole class is an excellent way to help students begin to see that all measurements are approximate. One can fine-tune measurements to get a degree of precision in a particular situation, but no matter how precise the instruments with which a measurement is made, error is alway possible.

The Investigations in *Covering and Surrounding* primarily involve whole-number situations as students begin to develop methods for finding area and perimeter. But they will also encounter many situations with rational numbers. In addition, students are likely to need fractions or decimals when measuring real objects. Students, especially those uncomforatble with fractions or decimals, may try to round all measurements to whole numbers. You will need to encourage them to use fractions or decimals so that their measurements are more accurate.

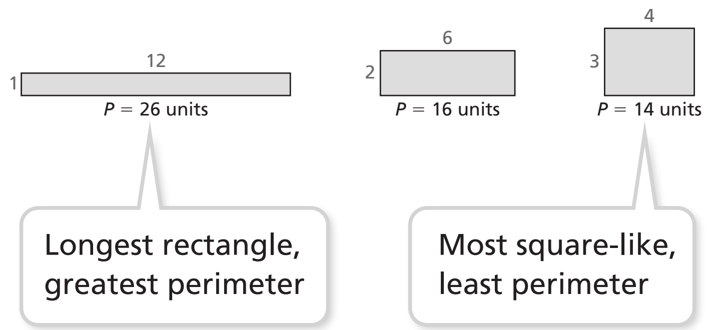
Measurement gives rise to proportional reasoning through measurement conversions. Because proportional reasoning is a key concept for the middle grades, it is important that students encounter many situations that call for reasoning about proportions. The familiarity of the situation can help students make sense of the relationships. Because they will be working with drawings that represent real objects, students will encounter problems of scale. We have tried to keep these problems manageable by carefully selecting scales that make for easy transitions from the model to the real object. The Investigations have a mix of metric and standard measures.

Students will often be asked to estimate or to compare measurements. Comparing and estimating are important skills used with many kinds of quantities. Students will need to know how to make estimates, whether an estimate is reasonable or appropriate, and how to compare measurements in meaningful ways.

The Relationship Between Shape and Size—Maximum and Minimum

It is important for students to explore how measurements are affected when an attribute such as area or perimeter is held constant. For example, some students hold the misconception that if you know the area of a shape, you can find the perimeter. Yet for any given area, you can make many different shapes, all with different perimeters.

For example, in a set of rectangles with whole-number dimensions and equal areas, the rectangle that has a shape most like a square has the least perimeter, and the rectangle that is the longest has the greatest perimeter. If the area is 12 square units, then the 1-by-12 rectangle has the *greatest* perimeter.



But if the dimensions are not restricted to whole numbers, then a $\frac{1}{2}$-by-24 rectangle has a greater perimeter and a $\frac{1}{4}$-by-48 rectangle has an even greater perimeter.

The same is true when perimeter is held constant. For a given perimeter, there are many different shapes that can be designed, each with an area that depends on the shape itself.

Fixed Area

Suppose the area of a rectangle is 24 square units and its dimensions are restricted to whole numbers. The rectangle with the least perimeter is the 4-by-6 rectangle. In the set of real numbers, the rectangle with the least perimeter is the square whose side lengths are $\sqrt{24}$. This will be explored in the *Looking for Pythagoras* Unit in Grade 8, after students have been introduced to square roots and irrational numbers.

If the 24-square-unit shape can be something other than a rectangle (and if it can have flexible walls), then the best design is a circle with radius of approximately 2.76 meters. The circumference (perimeter) of this circle is approximately 17.37 meters.

Fixed Perimeter

Suppose the perimeter of a rectangle is 24 inches and its dimensions are restricted to whole numbers. The rectangle that has the greatest area is the 6-by-6 rectangle. The 1-by-11 rectangle has the least area. If the dimensions are any real numbers, then there is no rectangle that has the least area. The graph of length versus area has the shape of a parabola. This relationship will be revisited in the algebra unit *Frogs, Fleas, and Painted Cubes.*

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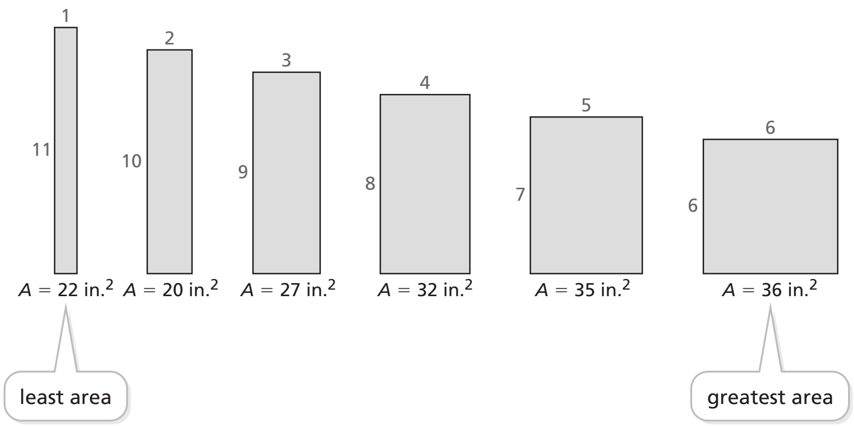
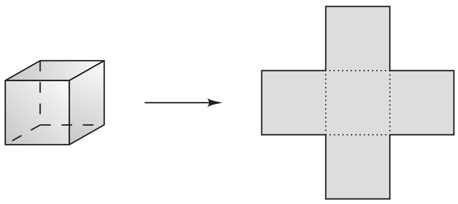


In Investigation 1, and in other places throughout the Unit, students will have an opportunity to explore the relationship between shape and size. They will consider situations in which area is held constant and perimeter varies, as well as situations in which perimeter is held constant and area varies. These Investigations help develop understanding of area and perimeter. The relationship between size and shape will be revisited when students study volume and surface area in the Grade 7 Unit *Filling and Wrapping*. This work also provides a foundation for future studies in calculus.

Surface Area and Volume

The surface area of an object is the total area of its faces. In Investigation 4, students are presented with the idea that an object is an empty box that can be unfolded into a flat surface. This new flat object is referred to as a “net.”



The area of the net corresponds to the surface area of the object. For some common figures, such as prisms and pyramids, the net will consist of shapes that can be easily broken apart into shapes such as triangles and rectangles.

Some students may have difficulties with picturing this folding and unfolding from the three-dimensional object to the two-dimensional net. Providing students with opportunities to build, deconstruct, and predict what the net or object will look like will help students improve their spatial visualization skills.

Students often have trouble distinguishing perimeter and area, and the differences between surface area and volume can be similarly confusing. These are the major differences:

Surface Area	Volume
Two-dimensional attribute	Three-dimensional attribute
Describes an object's surface	Describes the amount of space within an object
Standard unit is square unit	Standard unit is cubic unit
Sample units: in.^{2}, cm^2, ft^2	Sample units: in.^{3}, cm^3, ft^3

The cube works well to measure volume, because it easily makes stacks and layers, and all of its sides have the same length. However, any three-dimensional object can be used to measure volume as long as it can completely fill up the required space without leaving any gaps. In the metric system, 1 cubic centimeter is the same as 1 milliliter. This equivalence is convenient for measuring solid objects (with cubes) or the capacity of containers using a liquid measure in milliliters.

Students begin this Unit knowing the formula that Volume = length × width × height, but this formula applies only to right rectangular prisms. In this Unit, volume of rectangular prisms is revisited. First, students study prisms with side lengths that are whole numbers, and then their study is extended to prisms with fractional side lengths.

From their understanding of area of polygons, students develop an understanding of surface area of prisms. They establish a method for computing surface area of prisms by starting with a net that can be folded into a prism. This allows them to find the surface area of the prism as a two-dimensional shape. The folding of a net helps them visualize the surface area of a prism as a three-dimensional object. Volume of prisms is developed in *Filling and Wrapping*.

Area and Circumference of Circles

This Unit does not address the area of a circle since the Common Core State Standards (CCSS) place it in Grade 7. However, area of a circle is very appropriate for Grade 6. Contrasting area of polygons and area of circles provides a deeper undersanding of each. The area of rectangles is used as a unifying theme for developing area of polygons and circles. Students will find the area of the circle by using a transparent grid and counting the units needed to cover the circle. By embedding the circle in a square, students will notice that the area is approximately $\frac{3}{4}$ of d^2. At first, we leave the discussion as an estimate. However, one could write $\sim\frac{3}{4}d^2$ as $\sim\frac{3}{4}(2r)^2$ or $\sim\frac{3}{4}(4r^2)$ or $\sim 3r^2$ or πr^2. This method can lead to the more standard formula for the area of a circle.

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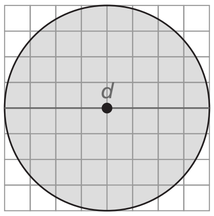
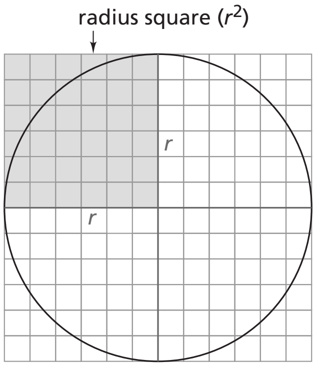


Students examine other estimating strategies, such as inscribing a square inside the circle and then averaging the areas of the circumscribed square and inscribed square to find the area of the circle. In addition to estimating techniques, this problem carries forward the unifying role of the rectangle in developing the area of triangles, parallelograms, and circles.

In the seventh-grade Unit *Filling and Wrapping*, the formula for the area of a circle is developed by finding the number of squares, whose side lengths are equal to the radius, that cover the circle. In the diagram, the circle is enclosed in a square. Two perpendicular diameters are drawn, which makes four squares whose areas are r^2. The area of the circle is less than $4r^2$. Then, either by finding the number of radius squares that cover the circle or by counting the area outside the circle but inside the larger square, the area of the circle is approximately 3 radius squares [$\pi(r \times r)$ or πr^2].



The circumference is found by counting the number of radius lengths needed to surround the circle. The number is about 3. The circumference of a circle is πd or $2\pi r$.