Mathematics Background

Extending Prior Knowledge to Compute With Decimals

In earlier *Connected Mathematics* Units, students explored meanings of and models for rational numbers, both in fraction form and in decimal form. Students in the middle grades solidify their understanding of number concepts by working with rational numbers and their various forms of representation and interpretation.

They have developed efficient algorithms for adding, subtracting, multiplying, and dividing with fractions. Students will build upon these algorithms to help them understand operations with decimals. In *Decimal Ops*, students will focus, in part, on understanding decimals as special fractions with denominators that are powers of 10. They use this information to form algorithms for the four main operations with decimals, such as in the example below.

Example

How might you complete this work to find the quotient 4.2 ÷ 0.84?

$$4.2 \div 0.84 = \frac{\blacksquare}{100} \div \frac{84}{100}$$
$$= \frac{\blacksquare}{84}$$
$$= \blacksquare$$

3.3 How Many Times? Dividing Decimals I

In elementary school, many students become familiar with place-value charts that include digits to the right of the decimal point. Grade 6 students may still be uncomfortable with decimal numbers, however. When students use only a place-value interpretation of decimal numbers, they must rely on number patterns to make sense of operations, especially the operations of multiplication and division. In *Decimal Ops*, students develop algorithms for operations with decimals using both fraction interpretations and place-value interpretations.

Helping Make Sense of Decimals

In this Unit, students will build algorithms for the four basic operations with decimals. Part of understanding the algorithms is recognizing when a situation calls for a particular operation. For example, students will explore the following Focus Question in Problem 1.1.

How do you decide which operation to use to solve a real-world problem?

This kind of thinking and reasoning supports the development of skill with the algorithms.

In Investigation 1 of *Decimal Ops*, students use their familiarity with money to make sense of operations with decimals. Throughout the Unit, students use money concepts and other decimal measures to help them build skill in working with decimals.

Developing Meanings of and Models for Decimals Through Connections to Fractions

Interpretations of Fractions

In *Decimal Ops*, students rely on their understanding and fluency with fractions to work with decimals. Fraction concepts have been explored and internalized in *Comparing Bits and Pieces* and *Let's Be Rational*. As they worked through those Units, students encountered these interpretations of fractions.

1. fractions as parts of a whole or set

Example

In *Comparing Bits and Pieces*, students use part-to-whole ratios to solve problems. The first step in these types of problems is identifying the pertinent part as a fraction of a whole.

Caleb and Isaiah are brothers. They share a 14-segment chewy fruit worm according to their age. How old could they be?

Comparing Bits and Pieces
2.2 Unequal Shares: Using Ratios and Fractions

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2. fractions as measures or quantities

Examples

Jasmine's raspberry jam recipe calls for $4\frac{1}{3}$ quarts of raspberries. She has picked $3\frac{1}{2}$ quarts of raspberries. About how many more quarts of raspberries should she pick?

Let's Be Rational
1.2 Estimating Sums and Differences

Mr. Flansburgh buys a $2\frac{1}{2}$-pound block of cheese. His family eats $\frac{1}{3}$ of the block. How much cheese has Mr. Flansburgh's family eaten?

Let's Be Rational
2.2 Modeling Multiplication Situations

3. fractions indicating division

Example

When you do the division 12 ÷ 5, what does the answer mean?

The answer should tell you how many fives are in 12 wholes. Because a whole number of fives will not fit into 12, you might write

$$12 \div 5 = 2\frac{2}{5}$$

Let's Be Rational
Investigation 3 Dividing With Fractions

These interpretations of fractions are helpful when working with decimals. The Grade 6 Units foreshadow interpretations such as fractions as operators ("stretchers" or "shrinkers") and fractions as rates, ratios, or parts of a proportion. These interpretations are explored more fully in later grades.

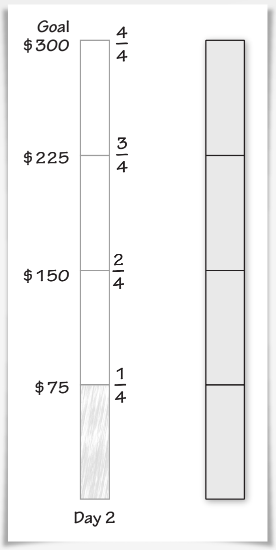
Models of Fractions

Connected Mathematics introduces models of rational numbers that connect directly to the important interpretations of rational numbers. The following models develop meanings of both fractions and their operations, which leads to fluency in computing with fractions.

1. fraction-strip models

Example

Did Mary label the dollar amounts on the fraction strip correctly?



Comparing Bits and Pieces
1.4 Measuring Progress: Finding Fractional Parts

2. number-line models

Example

What decimal number is halfway between 0.8 and 0.9?

Comparing Bits and Pieces
3.4 Decimals on the Number Line

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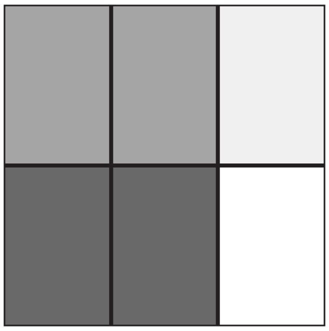
3. grid-area models

Example

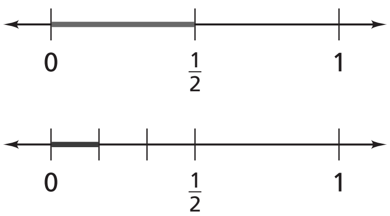
Mr. Sims asks to buy $\frac{1}{2}$ of a pan that is $\frac{2}{3}$ full. What fraction of a whole pan does Mr. Sims buy?



4. partition models

Example

Seth is running $\frac{1}{3}$ of a $\frac{1}{2}$-mile relay race. How far will he run?



Similar models are used in *Decimal Ops*. For a more complete discussion of these models, please refer to the Teacher Guides for **Comparing Bits and Pieces** and **Let's Be Rational.**

Decimal Multiplication and Division

A number of key ideas that were explored while developing algorithms for fractions are continued in *Decimal Ops.*

Many students enter Grade 6 thinking that multiplication of two numbers always results in a product that is greater than both of the numbers. Their experience with whole-number multiplication has firmly established this incorrect belief.

In fact, multiplication of numbers that are not whole numbers may be interpreted as a "stretch" (resulting in a greater product) or "shrink" (resulting in a lesser product) depending on whether the rational number is greater than or less than 1.

Elementary Experiences	Grade 6 Experiences
$6 \times 8 = 48$ (stretch)	$6 \times \frac{1}{8} = \frac{3}{4}$ (shrink)
$70 \times 2 = 140$ (stretch)	$70 \times 60.2 = 434$ (stretch)
$156 \times 12 = 1{,}872$ (stretch)	$0.2 \times 12 = 2.4$ (shrink)

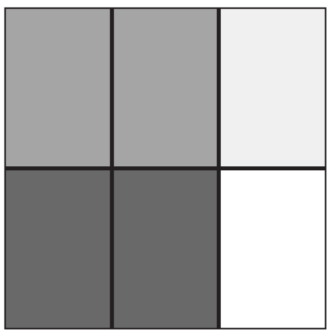
This concept will be developed more fully in the Grade 7 Unit *Stretching and Shrinking*.

Division also has its share of conceptual difficulties. The quotient of two rational numbers, whether in fraction or decimal form, is not necessarily less than the dividend. Again, the quotient depends on the size of the divisor. In elementary school, students are sometimes taught that quotients are less than the dividend, but this is not always true. What matters is whether the divisor is greater than or less than 1.

Elementary Experiences	Grade 6 Experiences
$12 \div 6 = 2$ (shrink)	$12 \div \frac{2}{5} = 30$ (stretch)
$55 \div 5 = 11$ (shrink)	$55 \div 0.25 = 220$ (stretch)
$96 \times 4 = 24$ (shrink)	$14.25 \div 4 = 3.5625$ (shrink)

The models that students used to multiply and divide fractions continue to be useful with decimals.

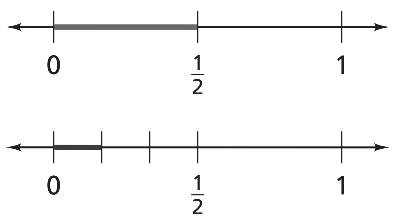
Grid Area Model



Represents both

$\frac{1}{2} \times \frac{2}{3}$ and $0.5 \times 0.\overline{6}$

Partition Model



Represents both

$\frac{1}{2} \div 3$ and $0.5 \div 3$

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Decimal Estimation

In *Comparing Bits and Pieces*, students worked with benchmark fractions and their decimal equivalents. These fraction benchmarks can be used to estimate decimal computations, as well. For example, estimations of the sum, the difference, the product, and the quotient of 0.78 and 0.14 are below. With each operation, students find estimating with benchmark fractions helpful.

Estimating Using Benchmark Fractions

0.78 + 0.14 is about $\frac{3}{4} + \frac{1}{8}$, or $\frac{6}{8} + \frac{1}{8}$.
A reasonable estimate is a little less than 1, or about 0.9.

0.78 − 0.14 is about $\frac{6}{8} - \frac{1}{8}$, or $\frac{5}{8}$.
A reasonable estimate is about 0.6.

0.78 × 0.14 is about $\frac{6}{8} \times \frac{1}{8}$, or $\frac{6}{64}$.
A reasonable estimate is about 0.1.

0.78 ÷ 0.14 is about $\frac{6}{8} \div \frac{1}{8}$.
A reasonable estimate is about 6.

Alternatively, using the same numbers as above, students can round to convenient decimals in their heads. They can estimate using the rounded, one-digit decimals. For example, a student might round 0.78 to 0.8 and 0.14 to 0.1. One decimal number is rounded up, and the other is rounded down. Students can estimate using the rounded numbers.

Estimating With Simpler Decimals

0.78 + 0.14 is about, but slightly greater than, 0.8 + 0.1, or 0.9.
0.78 − 0.14 is about, but slightly less than, 0.8 − 0.1, or 0.7.
0.78 × 0.14 is about, but slightly greater than, 0.8 × 0.1.
So, the product is close to 0.1.
0.78 ÷ 0.14 is about 0.8 ÷ 0.1, or 8.

Since the dividend was rounded up and the divisor was rounded down, the estimate will be greater than the actual answer. So, a good estimate is 6.

There is no single right way to estimate that works in every situation. Through experience, discussion, and analysis of what works in given situations, students need to build a repertoire of strategies for estimation.

Developing Algorithms for Computing With Decimals

Students have two ways to make sense of decimals:

1. Students extend the place-value system to the right of the decimal point.
2. Students interpret decimals as fractions.

These two concepts are related, but they have a different look and feel to students. In order to have the most robust understanding of decimals, and skill with computation of decimals, students need to understand each of these meanings and be able to apply them.

Depending on the operation and situation at hand, students might find one of the interpretations to be easier or more efficient. One method, either using fractions or extending the place-value system, may lead more quickly to an algorithm for working with decimals. Students need to look at the algorithms developed through each lens, however, in order to develop a deep understanding of operations with decimals. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.

Decomposing a Decimal Number

$$14.809$$

$$=$$

Decimal expansion

$$10 + 4 + 0.8 + 0.00 + 0.009$$

$$=$$

Fraction expansion

$$\frac{10}{1} + \frac{4}{1} + \frac{8}{10} + \frac{0}{100} + \frac{9}{1{,}000}$$

As seen in the video above, decimals can be decomposed into sums of their digits' values, which can then be converted to equivalent fractions. They can use the fraction expansion and knowledge of operations with fractions to form algorithms for operations with decimals. Note that the decimal 14.809 can be represented by the expression $\frac{10}{1} + \frac{4}{1} + \frac{8}{10} + \frac{0}{100} + \frac{9}{1{,}000}$, which can be rewritten as $\frac{10{,}000}{1{,}000} + \frac{4{,}000}{1{,}000} + \frac{800}{1{,}000} + \frac{0}{1{,}000} + \frac{9}{1{,}000}$. This helps students identify the fraction equivalent of 14.809, $\frac{14{,}809}{1{,}000}$. Students can use decimal and fraction expansions to develop algorithms for decimal operations.

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Extending the Place-Value System

The location of a digit in a number gives information about the value of the number. This is an important concept for students to understand. Without place-value understanding, students will find it very difficult to work with decimals.

Students use place-value number sense in Investigation 1 to analyze sums that are clearly incorrect, such as in Problem 1.2.

> Tat Ming shops for party food. He wants to buy the following items:
>
> 
>
>
> The cashier told Tat Ming that his bill was $34.75. What error do you think the cashier made in ringing up the bill?
>
> **1.2** Getting Close: Estimating Decimal Calculations

This analysis is continued in Investigation 2. Students practice adding decimals by examining the place of each digit. They recognize that they must be careful to add digits in the same location with respect to the decimal point.

> Four students calculated the sum 2.561 + 15.74 + 92.03 and got different answers. Study their work to see if any of their sums are correct. For those that are not correct, identify the errors they made.
>
> **2.1** Getting Things in the Right Place: Adding Decimals

In *Decimal Ops*, students also look at the patterns in problems such as in Problem 3.1, which uses a place-value perspective to develop an algorithm for multiplication of decimals.

> Use the fact that $21 \times 11 = 231$. Find each product.
>
> **1.** 2.1×11 **2.** 2.1×1.1 **3.** 2.1×0.11
> **4.** 2.1×0.011 **5.** 0.21×11 **6.** 0.021×1.1
> **7.** 0.021×0.11 **8.** 0.21×0.011 **9.** 0.021×0.011
>
> **3.1** It's Decimal Time(s): Multiplying Decimals I

The same perspective is used to help students develop a division algorithm.

Decimals as Fractions With Denominators of Powers of Ten

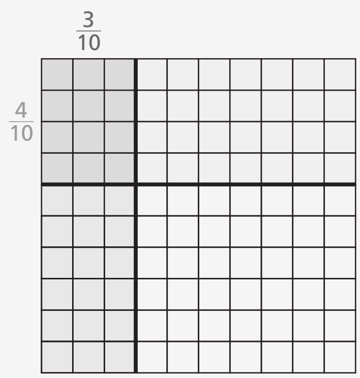
When computing with decimals, students can also consider the fractional equivalents of the decimals. For fractions with denominators of powers of ten, the denominator of a fraction gives the same information about the value of a number as the location of a digit in a decimal number.

In *Decimal Ops*, students first decompose decimals into sums of fractions where each fraction represents a single digit.

Investigation 2 ACE Exercise 3

They then use models from their previous work with fractions to represent decimal products. Since students have prior experience with multiplying fractions within these models, they can use their knowledge to see patterns in where decimal points are placed in products of two decimal numbers.

You can write 0.4×0.3 as $\frac{4}{10} \times \frac{3}{10}$. The area model below shows how to find the fractional answer of this product.



What decimal is represented by this area model?

3.1 It's Decimal Time(s): Multiplying Decimals I

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Last, by converting decimals to fractions with common denominators, students can use their knowledge of fraction division to identify the proper placement of the decimal point in decimal division.

Example

How might you complete this work to find the quotient 4.2 ÷ 0.84?

$$4.2 \div 0.84 = \frac{\blacksquare}{100} \div \frac{84}{100}$$
$$= \frac{\blacksquare}{84}$$
$$= \blacksquare$$

3.3 How Many Times? Dividing Decimals I

Decimal Forms of Rational Numbers

Students may have already observed that the decimal forms for some fractions, such as $\frac{1}{3} = 0.3333333\ldots$, continue indefinitely but display a repeating pattern.

Some infinite decimals, such as 0.10110111011110111110 . . ., never reach a point at which the digits repeat. These are irrational numbers. Other examples of irrational numbers are π and $\sqrt{2}$. These irrational numbers are not explored in this Unit. They are discussed in the Grade 8 Unit Looking for Pythagoras. In *Decimal Ops*, students work with the decimal forms of rational numbers.

Some rational numbers have a finite (or terminating) decimal form.

Example

$\frac{1}{2} = 0.5$
$\frac{3}{4} = 0.75$
$\frac{1}{8} = 0.125$
$\frac{3}{25} = 0.12$

Others have an infinite repeating decimal form.

Example

$\frac{2}{3} = 0.666666666\ldots$
$\frac{8}{15} = 0.5333333$
$\frac{3}{7} = 0.42857142857142\ldots$

Some rational numbers in simplified fraction form have only 2's or 5's in the prime factorization of the denominator. Only these numbers will have a terminating decimal form.

For example, $\frac{12}{75} = 0.16$, in simplified fraction form, is $\frac{12}{75} = \frac{4}{25}$. The denominator, 25, has only factors of five in the denominator (5×5). So, this number terminates.

On the other hand, fractions such as $\frac{13}{75}$ and $\frac{4}{3}$ do not terminate. The denominators in these fractions, 75 and 3, have factors besides just 2 and 5 ($75 = 3 \times 5 \times 5$; and $3 = 3$). These decimal forms repeat: $\frac{13}{75} = 0.17333333\ldots$ and $\frac{4}{3} = 1.333333\ldots$

Percents

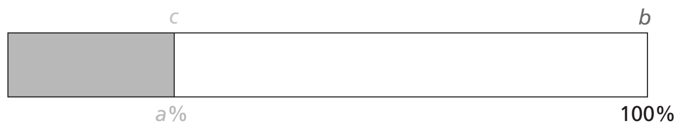
Using Percent Bars as Diagrams

In *Decimal Ops*, students use their knowledge of operations with decimals to further develop understanding and skill in solving percent problems.

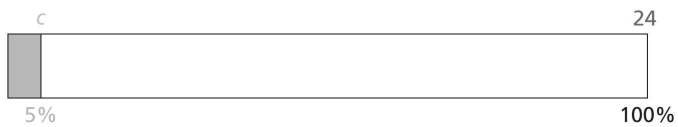
In *Comparing Bits and Pieces*, students used percent bars, or tape diagrams, to find what percent one number is of another. In *Decimal Ops*, students work with a greater variety of percent problems. For a problem that states that $a\%$ of $b = c$, students work to find a missing variable when any of the three values a, b, or c is missing.

In many programs, the three cases for the missing variable are taught as separate, unrelated problems. Instead, students should see that, no matter which variable they are tasked with finding, all are versions of this basic relationship.

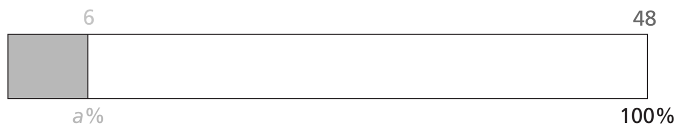
Using $a\%$ of $b = c$, students can find any missing variable by using a percent bar.



For example, for 5% of $24 = c$, students can use the percent bar below.



To find the value of a in the example $a\%$ of 48 equals 6, students can use the following percent bar.



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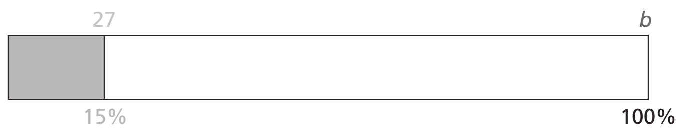
To find an original number, such as *b* in the example 15% of *b* = 27, the following percent bar organizes the three related quantities.



These diagrams help students visualize the relationship between the three numbers. They also assist students in writing appropriate related number sentences.

Problem Solving With Percents

Students solve a variety of problems in this Unit. For each problem, students can use the number sentence *a*% of *b* = *c*.

> Jill wants to buy an album of music that is priced at $7.50. The sales tax is 6%. What will be the total cost of the album?
>
> **4.1** What's the Tax on This Item?

In this Question, students know the price of the item and the sales tax. They need to find 6% of $7.50. They are tasked with finding the value of *c*.

Students can use the number sentence 6% of $7.50 = *c* to solve the problem.

They have to add together *b* and *c*, however, to find the final answer, the total cost of the album.

> A customer left Jerome $2.50 as a tip for service. That tip is 20% of the total bill for the food. How much is the bill?
>
> **4.2** Computing Tips

For this Question, students can write the number sentence 20% of *b* = $2.50. They have to find the value of *b*. Students can solve this number sentence in several ways:

> It takes five 20%'s to make 100%.
> Since 20% is $ 2.50, we need 5 $ 2.50's to find 100%.
> 5 × $ 2.50 = $ 12.50, so the bill must have been $ 12.50.

Later in *Connected Mathematics*, students will develop more sophisticated equation-solving techniques. They will be able to solve a problem like this by using inverse operations:

$$20\% \text{ of } b = \$2.50$$

$$\frac{0.2 \times b}{0.2} = \frac{\$2.50}{0.2}$$

$$b = \$12.5$$

At a music store, Rita got a $12 discount on an item originally priced at $48. What percent discount did she get?

4.3 Percent Discounts

In this situation, students need to find the value of a in the number sentence $a\%$ of $\$48 = \$12$. Again, students can use a few different methods.

I need to find what percent 12 is of 48.

I know that 12 is $\frac{1}{4}$ of 48.

$\frac{1}{4}$ of 100% is 25%, so 12 must be 25% of 48.

As with the previous example, students will develop more sophisticated solution methods later in the *Connected Mathematics* curriculum. They will be able to write the equation $48a = 12$ from the number sentence $a\%$ of $\$48 = \$12$. They can divide each side of the equation by 48 to find that $a = 0.25$, or 25%.

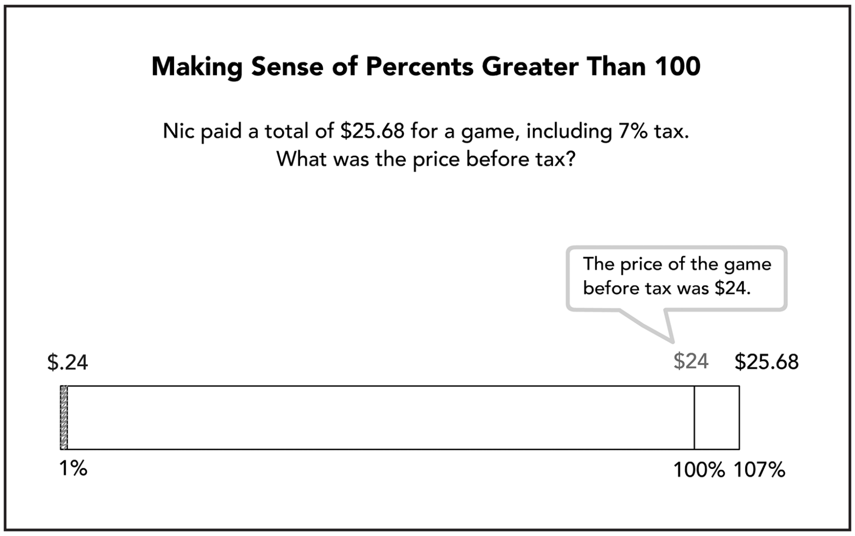
The informal techniques that students use before directly learning about inverse operations are powerful. Students who use these techniques demonstrate deep understanding of the situation. Students who understand these concepts will be better equipped to self-monitor their work when they begin to more fully develop equation-solving techniques in Grades 7 and 8. If students are encouraged to use the more complex techniques too early, they may mask misunderstanding of the problem situations and what the problem is asking.

Working Backward With Percents

Students are introduced to another type of percent situation, which the Question below demonstrates:

> Nic paid a total of $25.68 for a game, including 7% tax. What was the price before tax?
>
> **4.1** What's the Tax on This Item?

In this scenario, students are given the final total of a purchase: the price plus the tax. They start with a percentage greater than 100%. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.



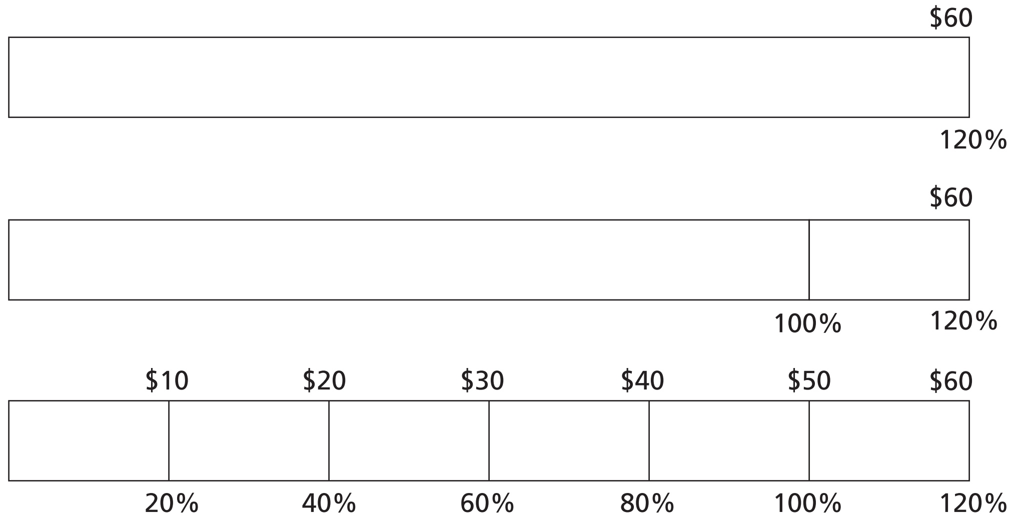
This can be further explained with a real-life scenario. A person may enter a restaurant with a fixed amount of money. This person knows what percent of the bill will be added on for taxes and tip. The person needs to figure out how much he can spend on food and still pay the bill. Consider the following example.

> *Chandra and her friends have $60 to spend at a pizza parlor. The tax on food is 5% and Chandra's group wants to leave a 15% tip on the food before tax. How much money can Chandra's group spend on food before tax and tip?*

For this example, students can simplify the problem if they realize that they can add together the 5% tax and the 15% tip to make a total of 20% extra cost. Students can simplify in this way because the percents are of the same number ($0.15x + 0.05x + 0.20x$). If the tip were also on the tax, the situation would become more complicated.

Students can use percent bars to model the problem:

Percent Bars for 120%



Since the tax and tip are 20% of the cost of the meal, the total cost can be thought of as 120%. The cost of the original bill is represented by 100%. Students can partition the 120% bar into six equal parts, each of which is 20%. The six partitions show us that each 20% of the total cost represents $10. So the cost of the food ordered must not exceed $50, or 100%.