Mathematics Background

Patterns of Change

Through their work in *Variables and Patterns*, your students will learn that a variable is a quantity that might assume many different values. They will work on problems that require them to predict the pattern of change in a variable as time passes or to predict the way changes in values of one variable are related to changes in values of another.

This Unit introduces some basic tools of algebra. Students are not expected to develop a complete understanding of algebraic ideas. These ideas will be revisited and further developed in each of the succeeding algebra units.

Verbal Descriptions

Verbal descriptions of a relationship are useful because they are descriptions in students' everyday language. This helps students form mental pictures of the situations and the relationships among the variables. The disadvantage of verbal descriptions is that they are sometimes ambiguous, making it difficult to get a quick overview of the situation and the relationships among variables.

Tables

Tables are usually easy to read. From a table, you can see how a change in one variable affects the value of the other variable. Students can recognize whether the change is additive, multiplicative, or unpredictable. Once students recognize the pattern of change, they can extend the pattern to get the next entry.

Example

Consider these tables.

Table 1: Linear Relationship

x	0	1	2	3
y	5	6	7	8

Table 2: Exponential Relationship

x	0	1	2	3
y	1	2	4	8

In Table 1, as the values of the variable x change by 1 unit, the values of y change by 1 unit. The table can be continued by adding 1 to the previous entry in the x column and 1 to the previous entry in the y column. If x is 4, then y is $8 + 1$, or 9. The table can be extended backward by reversing the pattern of change. If x is -1, then y is 5 minus 1, or 4. This pattern of change is characteristic of a linear relationship.

The change pattern in Table 2 is characteristic of exponential relationships. It is a multiplicative pattern because the values of y double, or increase by a factor of 2, as the values of x increase by 1 unit.

In some tables, the patterns of change are not regular.

Example

Consider Table 3, which occurs in the first Investigation. It does not show a pattern of change that is regular; that is, there is no way to predict the change from one point to the next.

Table 3: Unpredictable Change

Time (h)	Distance (mi)
0.0	0
0.5	8
1.0	15
1.5	19
2.0	25
2.5	27
3.0	31
3.5	38
4.0	40
4.5	40
5.0	45

Graphs

Graphs are another way to view relationships and patterns of change between variables. Graphs 1 and 2 can be thought of as pictures of the relationships in Tables 1 and 2.

Graph 1

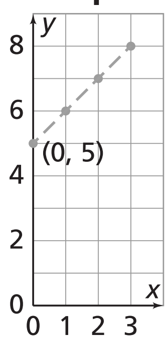

Table 1

x	y
0	5
1	6
2	7
3	8

The linear relationship, which has a constant rate of change, is represented by a straight-line graph. This relationship can be represented symbolically as $y = x + 5$.

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Graph 2

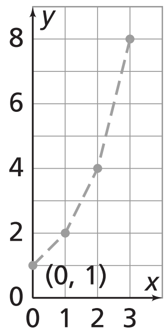


Table 2

x	y
0	1
1	2
2	4
3	8

The exponential relationship, which has a multiplicative rate of change, is characterized by a curve. This relationship can be represented symbolically as $y = 2^x$.

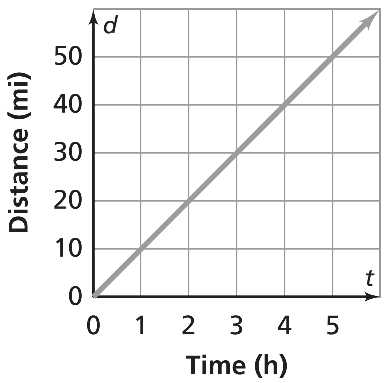
Linear relationships are the focus of the Grade 7 Unit *Moving Straight Ahead*, and exponential relationships are studied in the Grade 8 Unit *Growing, Growing, Growing*.

Variables and Patterns provides a foundation to study important patterns of change. Only simple linear equations are explored in this Unit.

Example

For example, the distance *d* a cyclist can cover depends on time *t* and the rate *r* at which the cyclist pedals. If a cyclist rides at a steady rate of 10 miles per hour, then $d = 10t$. This is a linear relation—its graph (below) is a straight line.

Distance Covered by Cyclist



The shape of a graph of a pattern of change over time shows the rate of change in the dependent variable as time passes. A straight-line graph indicates change at a constant rate. A curved graph indicates change at a variable rate.

Note that the rate of change continues the rate concepts that were introduced in *Comparing Bits and Pieces*. The cyclist bikes at 10 miles per hour. This is a unit rate. It is the coefficient of the independent variable and determines the steepness of the line. In *Moving Straight Ahead*, students learn that this is also called the slope of the line.

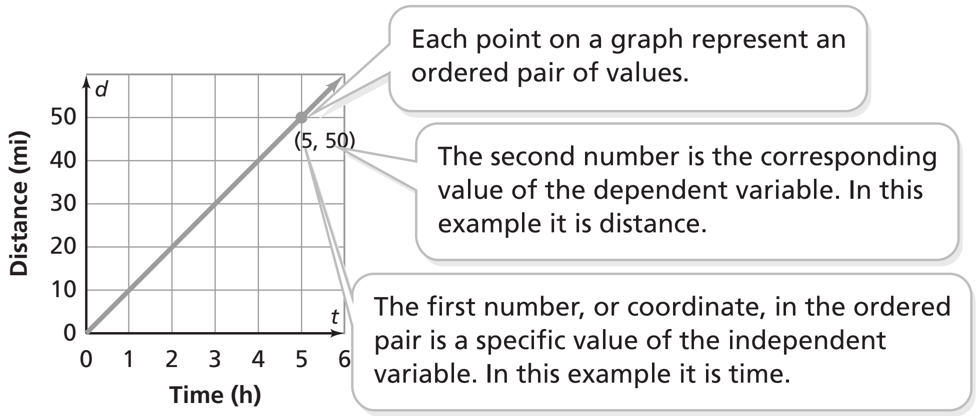
Other graphs show correlations between variables in which time is not the independent variable. The independent variable is some other manipulated variable that produces correlated change in a responding variable. An example is the profit changing as the number of books sold increases.

Graphs are more abstract than tables for many students. Therefore, the Problems in this Unit address with some care the issues that arise in constructing graphs of relationships between variables and the interpretation of the resulting pictures.

The Problems provide specific guidance on the basic steps required in graph construction, and sort out issues about ways it might make sense to connect the known data points. Students go through the steps of graph construction and interpretation, using given data as well as data they collect from a jumping-jack experiment.

Construction and analysis of graphs of quantitative relationships are tasks that students will meet again and again as they move through middle school, high school, and college mathematics. This Unit begins building the understanding and skill required by those tasks.

It is important to focus on two key ideas about graphs. The first is that each point on a graph represents an ordered pair of values. The second key idea is that the collection of points on a graph tells a story of how the values change.



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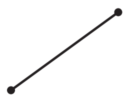
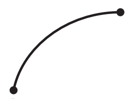
The collection of points on a graph tells a story of how changes in one variable are related to changes in the other. The story might be one of change over time or of one variable's change in response to change in another.

In the examples below, the points representing the time and distance of a cyclist can be connected in many ways. A straight segment connecting two points implies that the cyclist traveled at a constant rate in the time interval between the points. The four graphs below show other ways two points may be connected.



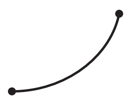
1

Graph 1 shows the cyclist riding at a steady rate for the whole time interval.



2

Graph 2 shows the cyclist starting fast and then gradually slowing down.



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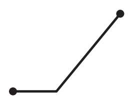
Graph 3 shows the cyclist starting slowly and then gradually speeding up.



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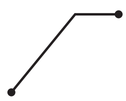
Graph 4 shows the cyclist not moving for the first part of the time interval and then riding at a steady rate for the rest.



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Graph 5 shows the cyclist riding at a steady rate for the first part of the time interval and stopped for the rest.

As students choose scales for the axes of a graph, plot data points, connect plotted data points, and interpret the overall shape of the graph, you should consistently probe their thinking with questions about why they make the choices they do and how they draw the conclusions they do. To keep the issues in mind, plan on asking the following questions throughout the unit:

- What is the shape of the graph?
- What choices did you make to construct the graph?
- How do you know you are right?

Discrete Versus Continuous Data

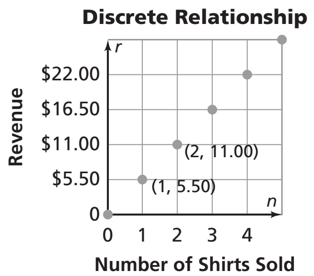
From a statistical perspective, there are two basic families of quantitative variables—those that can take on only a countable set of values (discrete data) and those that can take on essentially any real-number value in an interval (continuous data).

Tables can represent only discrete collections of (x, y) values; graphs can represent both discrete and continuous data.

Example

Many situations are discrete relationships, such as the number of sweatshirts sold and the revenue.

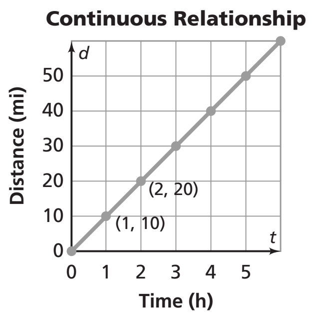
If the shirts sell for \$5.50, then the revenue r for selling n shirts is $r = 5.50n$. In this situation, it does not make sense to connect the points. Points (1, 5.50) and (2, 11) are on the graph; however, if these two points are connected it would imply that $1\frac{1}{2}$, or part of a shirt, could be sold.



Other situations, such as the distance-time-rate relation, are not discrete; they are continuous. Graphs can accurately display continuous data.

Example

For example, if a bicyclist pedals at a steady rate of 10 miles per hour, then distanced d after t hours is $d = 10t$. In the graph of $d = 10t$, it is reasonable to connect the points (1, 10) and (2, 20) since one can travel $1\frac{1}{2}$ hours and go a distance of 15 miles; it makes sense because time is a continuous quantity.



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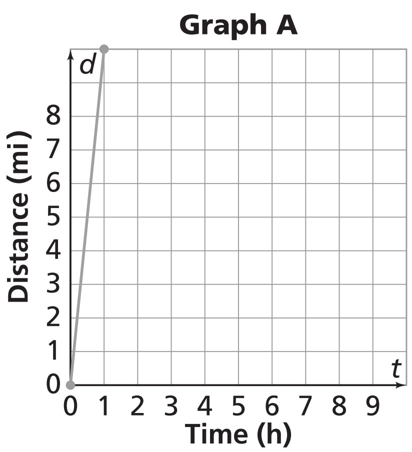
In many cases it is possible to use a continuous graph to represent relationships in which one (or both) of the variables is limited to a discrete domain or range. Such continuous graphs help suggest the pattern of the relationship and, when the variables are discrete but with fairly dense possible values, the continuous graph is not usually misleading.

When constructing tables or graphs of relationships between variables, there is some value in recognizing whether the variables involved are discrete or continuous. Students should ask themselves questions such as these:

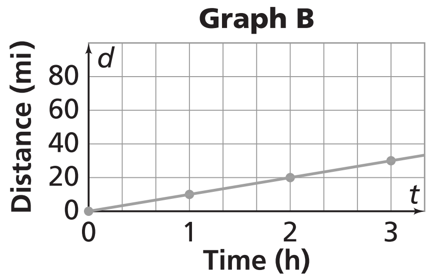
- What values make sense for the variables I am studying?
- Does it make sense to connect the points on the graph?
- Does it make sense to try to find values between those on the graph?

Selecting a Scale

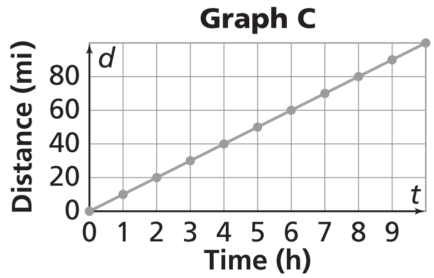
Another aspect of graphing is that of scale. This is closely connected to the range of values for each variable. To represent a relationship graphically, students must have a good feel for the range of values. Students must select an appropriate scale so that the relevant pieces of the graph can be displayed. The effects of the scale can often lead to distortion in the interpretation, as shown in the graphs below.



For example, suppose students select a scale of 0 to 10 for both axes when they graph the equation $d = 10t$, as in Graph A. Only the information about the first hour would be shown on this graph. This may not be enough information for students to understand the relationship.



The scale in Graph B may lead students to believe that the distance covered in three hours is minimal.



Graph C best represents the situation.

Equations

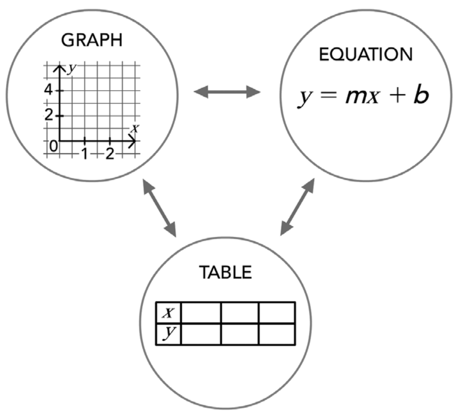
Equations, in addition to words, tables, and graphs, are another way to represent relationships between two quantitative variables. For the most part in this Unit, equations are of the form $y = mx + b$ or $y = mx$. The expression on the right side represents the dependent variable y.

In the third Investigation of this Unit, students develop strategies for writing symbolic expressions to represent simple relationships. The value of such expressions is that they are brief and represent a complete picture of the pattern, while tables and graphs can show only parts of the relationships.

While relationships between variables are the most important idea in this Unit, it is the representation of these relationships that is the dominant theme. It is important for students to move freely among the various representations. It may not be obvious initially to students how the entries in a table relate to points on a graph or to solutions of a symbolic equation; and conversely, how solutions to an equation or points on a graph relate to entries in the table. These connections, however, are explored in depth in this Unit. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.

Representations of Relationships

Problem Context = Verbal Description = Relationship in Words



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By the end of the Unit, students should feel very comfortable with tables and graphs and with some simple equations. Students should also have an appreciation of the advantages and disadvantages of each representation.

In this Unit, students write algebraic equations modeling a variety of contexts. They can use the words or clues in the context to represent the relationship as a symbolic statement or equation.

Problems typically involve a relationship between variables in which each value of the first variable is connected to one and only one value of the second, such as when there is a single distance that corresponds to each time of travel.

Most relationships can be defined by rules that tell how to find the value of the second variable associated with any given value of the first variable.

> **Example**
>
> For example, to find the distance traveled in h hours at a speed of 50 miles per hour, multiply h by 50. The same rule can be written with a symbolic algebraic equation as $d = 50h$.

Rules can generally be represented by equations in which the second variable appears alone on one side of the equation and an expression on the other side shows how to calculate the value of the second variable from the given value of the first variable.

Expression Versus Equation

Students are often unsure of the difference between the terms *equation* and *expression*. An equation is any mathematical sentence stating that two quantities or two expressions are equal in value. An expression is a sequence of variables, numbers, and operation symbols that can be evaluated when values of the variables are given. An expression does not contain any equal or order signs.

An expression can represent one of the variables or quantities in a relationship. The expression contains the other variable. For example, in the equation $I = 5x - 14$, the expression $5x - 14$ represents the variable y.

An equation is any mathematical sentence stating two quantities are equal in value.

$11x + 7$ $\qquad$ $y = 4x - 2$

An expression is a sequence of variables, numbers, and operation symbols that can be evaluated when values of the variables are given. An expression does not contain any equal or order signs.

In the context of ideas developed by Investigation 4, equations are used in two different ways. One use is to find the answer to a question.

> **Example**
>
> For example, $30 = 5x - 15$ is an equation in which the implicit direction is to find a value of x that makes the equation a true statement. Note that this equation can be thought of as an instance of the more general equation, $y = 5x - 15$.

The second use is that an equation expresses the rule for a relationship.

> **Example**
>
> For example, $P = 350n - 750$ relates P and n. The most common action with such an equation is to substitute a value for n in the expression $350n - 750$ to find the related value of P. One can also solve the equation for n given P.

In writing the Problems of Investigation 4, the authors have tried to be as clear as possible about the meaning of the technical terms as they are used. It will undoubtedly be helpful for teachers to frequently remind students of the different, but related, meanings of *equation* and *expression*.

Equivalent Expressions

One of the central ideas of algebra is the notion that any given symbolic expression can be written in a variety of equivalent forms. That is, the results of calculations prescribed by the two expressions are always the same.

> **Example**
>
> For example, the expressions $3x + 5x$ and $8x$ give the same value for any substituted value of x.

In this early algebra Unit we develop the basic meaning of equivalence and encourage students to check equivalence by comparing tables and graphs and by reasoning from problem context clues.

> **Example**
>
> For example, if each customer of Ocean Bike Tours brings in revenue of $350 and incurs operating costs of $250, then the profit from a tour with n customers is given by $350n - 250n$, or simply $100n$.
>
> The two expressions $350n - 250n$ and $100n$ are equivalent because they both make sense as models of the situation, because they produce identical tables and graphs of (*number, profit*) values, and because the Distributive Property assures their equivalence. The two expressions are equivalent ways to represent the dependent variable, revenue.

Order of Operations

When students begin writing algebraic expressions to represent relationships, they need to understand the conventions for evaluating algebraic expressions. Students will have been introduced to such Order of Operations rules in earlier grades and in *Prime Time*. However, the rules take on special importance when variables are involved.

In most numeric contexts, a person who writes down a calculation knows what he or she means and will seldom do the wrong calculation, even if what he or she writes is not perfectly true to the standard mathematical conventions.

When one writes algebraic expressions to represent calculating rules, however, one has to be careful with Order of Operations conventions because the rule has to work for any user and any numbers. Thus we revisit Order of Operations conventions in Problem 3.4 of *Variables and Patterns*.

One additional point worth making is that Order of Operations rules, such as *Do operations in parentheses first*, do not preclude writing a given expression in an alternate equivalent form before doing any calculations.

Example

It may be easier to evaluate $24\left(\frac{5}{8} + \frac{1}{3}\right)$ in the expanded form $24\left(\frac{5}{8}\right) + 8\left(\frac{1}{3}\right)$ than to find the sum $\left(\frac{5}{8} + \frac{1}{3}\right)$ before multiplying by 24. Similarly, it is often convenient to rearrange the order of terms in an expression to an equivalent form before carrying out operations from left to right. For example, $\frac{1}{3} + 2x + \frac{1}{6} + 4x$ is easierto simplify when you gather like terms first and write the expression as $2x + 4x + \frac{1}{3} + \frac{1}{6}$, which is equivalent to $6x + \frac{1}{2}$.

Solving Equations and Inequalities

There can be little doubt that solving equations is the most prominent task of elementary algebra. Most students spend months, if not years, learning a variety of techniques for solving linear, quadratic, exponential, radical, and rational equations.

Most students in sixth grade will already have done some solving of equations in problems that ask them to "find the missing number in an equation." They will encounter many more equations in standard algebraic form as they move ahead in *Connected Mathematics*.

At this point in the algebra strand we have four objectives with respect to solving equations and inequalities.

- Develop understanding that solving an equation involving variables means finding values of the variables that make the given equation a true statement.
- Develop understanding of strategies for solving equations by the guess-and-test method, enhanced by inspection of tables and graphs for related relationships.

  Example

  For example, to solve an equation such as $3x + 5 = 7$, one can study tables and graphs of values for the function $y = 3x + 5$ to find values of x for which $y = 7$.

- Develop understanding and skill in using fact families and inverse operation strategies for solving one-step equations.

  Example

  For example, to solve $3x = 7$, one can evaluate $x = 7 \div 3$. To solve $x + 3 = 7$, one need only evaluate $x = 7 - 3$, and so on.

- Develop understanding of what it means to solve a simple inequality and how to graph the solution on a number line graph.

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