Mathematics Background

Extending Understanding of Two-Dimensional Geometry

In Grade 6, area and perimeter were introduced to develop the ideas of measurement around and within polygons in *Covering and Surrounding*. This Unit focuses on polygons beyond triangles and quadrilaterals, developing the relationships between sides and angles. These relationships lead to such ideas as tessellations (tilings) of figures and reflection and rotation symmetry.

Attention is also given to the conditions needed to construct triangles and quadrilaterals. This leads to criteria for congruence of triangles, which is explored in the Grade 8 Unit *Butterflies, Pinwheels, and Wallpaper.*

Students also strengthen their measurement skills. Students make and defend conjectures that relate the sides and angles of a polygon. Students also model relationships using variables, a concept first seen in *Variables and Patterns*.

The development in *Shapes and Designs* is based on the van Hiele theory of geometry learning. We begin by building from students' experiences with recognition of shapes and classification of shapes in elementary grades. Then, we move on to analyzing the properties of those shapes. The overall development progresses from tactile and visual experiences to more general and abstract reasoning. We assume students have had some prior exposure to the basic shapes and their names.

Polygons

A simple polygon is a planar figure consisting of at least three points p_1, p_2,...p_n, called vertices, that are connected in order by line segments. These line segments are called sides (with point p_n connected to point p_1) so that no two sides intersect except as prescribed by the connection of consecutive vertices. Figures with pieces that are not line segments, figures that cannot be traced completely from any vertex back to that vertex, figures that do not lie in a single flat surface, and figures that have sides crossing at points other than vertices are not usually called polygons. Those distinctions are illustrated in the display of polygons and nonpolygons at the start of Investigation 1 in the Unit.

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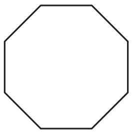
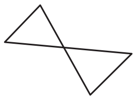
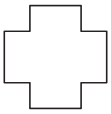
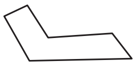
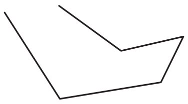
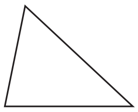
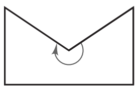
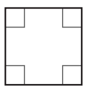
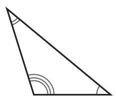
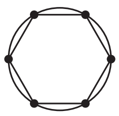
Examples of Polygons	Non-Examples
	
	
	

Polygons are generally named by the number of sides and described with several other special adjectives. Below is a summary of the types of polygons.

Term	Definition	Picture
Convex	All interior angles measure less than 180°.	
Concave	At least one angle measures greater than 180°	
Regular	All sides and angles equal.	
Irregular	Not all sides or angles are equal.	
Cyclic	All vertices lie on a single circle.	

In this Unit, we will focus on simple convex polygons.

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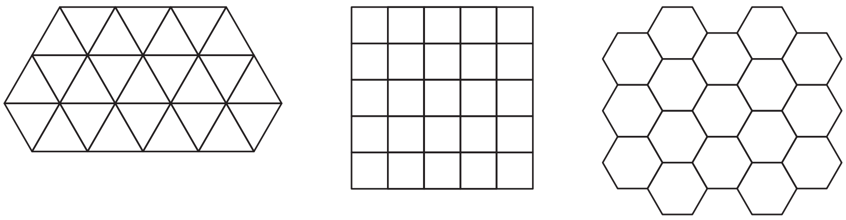
An important distinction to keep in mind in geometric units is that a polygon consists of only the line segments (or sides) that make up the polygon. These line segments enclose a region of the flat surface. This region is sometimes called the *interior of the polygon* or *polygonal region*. The points in the interior are not part of the polygon, and the points on the sides of the polygon are not part of the interior. We can also talk about the exterior region of a polygon—this is, the set of points that are neither on the polygon nor in the interior of the polygon. The distinctions that hold for polygons and polygonal regions also hold for any closed plane figure, including circles.

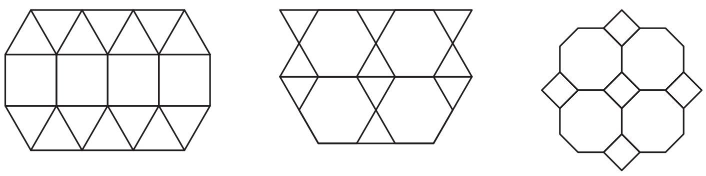
In the Grade 6 Unit *Covering and Surrounding*, the primary focus was on perimeter of the polygon and area of the polygonal region. Technically speaking, when we talk about area we should say "area of the rectangular region or triangular region," and so on, but it has become common practice to say "area of a rectangle." It is understood that this is the area of the interior of the rectangle or the area of the rectangular region created by the rectangle. The distinction between polygon and polygonal region is important to note so that students do not take away unintentional misconceptions from the work or discussion in class.

Tessellations

The first big question presented in *Shapes and Designs*, to motivate analysis of polygons, is the problem of tiling, or tessellating, a flat surface. The key is that among the regular polygons (polygons with all edges the same length and all angles the same measure), only equilateral triangles, squares, and regular hexagons will tile a plane.



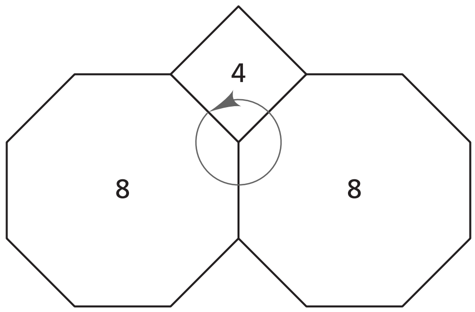
Many other figures, and combination of figures, can be used to tile a flat surface.



When one understands the important properties of simple polygons, one can create an abundance of aesthetically appealing tiling patterns, complete with artistic embellishments in the style of artist M. C. Escher. However, it is the discovery of the important properties of the figures that make the tiling possible, not the tiling question itself, that is one of the foci of the Unit.

For regular polygons to tile a flat surface, the angle measure of an interior angle must be a factor of 360. So, an equilateral triangle (60° angles), a square (90° angles), and a regular hexagon (120° angles) are the only three regular polygons that can tile a flat surface. Copies of each of these will fit exactly around a point in a flat surface (or plane).

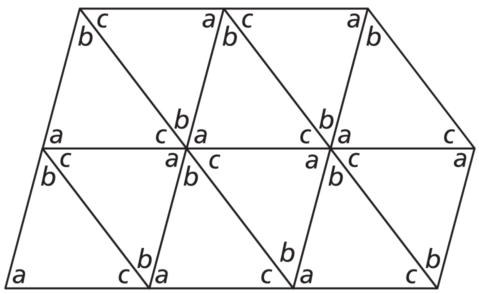
There are eight combinations of regular polygons that will tile. The numbers in parentheses refer to the polygon by side number—8 means a regular octagon, 6 means a regular hexagon, etc. The sequence of numbers represents the order they appear around a vertex of the tiling. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete image gallery for the example below.



2 octagons and 1 square

Note that there are two arrangements with triangles and squares, but depending on the arrangement they produce different tile patterns, so order is important.

In addition, any triangle or quadrilateral will tile a flat surface as in the examples below:



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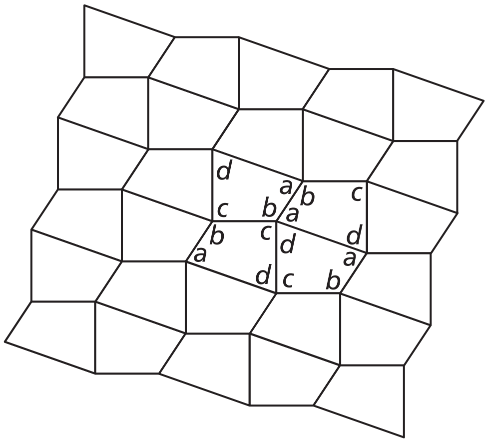
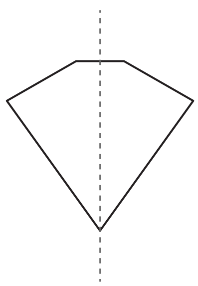


Symmetries of Shapes

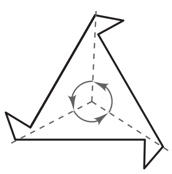
Among the most important properties of polygons are reflection and rotation symmetry. Reflection symmetry is also called mirror symmetry, since the half of the figure on one side of the line looks like it is being reflected in a mirror. A polygon with reflection symmetry has two halves that are mirror images of each other. If the polygon is folded over the line of symmetry, the two halves of the polygon match exactly.

Reflection Symmetry

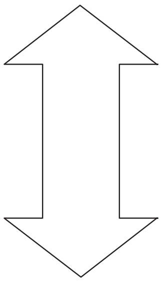


Rotation symmetry is also called turn symmetry, because you can turn the figure around its center point and produce the same image. All shapes have “trivial” rotation symmetry in the sense that they can be rotated 360° and look the same as before the rotation. When we determine whether or not a shape “has” rotation symmetry, we check for rotation symmetry for angles less than 360°.

Rotation Symmetry



However, the convention is that once we determine that a shape has rotation symmetry, when counting the rotation symmetries, we include that "trivial" rotation as well. For example, the shape below has 2 rotation symmetries: 180° and 360°.



This convention works out nicely because we can say that a square has four rotation symmetries, a regular pentagon has five rotation symmetries, and a regular hexagon has six rotation symmetries. In general, a regular polygon with *n* sides will have *n* rotation symmetries.

Angle Measures

The shape of a polygon is strongly linked to the measures of angles formed where its sides meet. One standard definition of *angle* is the union of two rays with a common endpoint. Any pair of adjacent sides in a polygon determines an angle if one imagines those sides extended without bound away from the common vertex. The concept of rotation symmetry leads to another way of thinking about angles as descriptors of turning motions—carrying one side of an angle onto the other. Both of these conceptions of the term *angle* are developed in this Unit. A third conception of *angle* as a region (like a sector of a circle or a piece of pizza) is introduced in the ACE exercises of the Unit.

In all contexts for thinking about angles, it is usually helpful to measure the figure or motion being studied. For students, it is important to have both informal "angle sense" and skill in use of standard angle measurement tools. Angle sense is developed in this Unit by starting from the intuitive notion that an angle of measure one degree is $\frac{1}{90}$ of a right angle or square corner or $\frac{1}{360}$ of a complete turn. Then, students develop familiarity with important benchmark angles (multiples of 30° and 45°) by playing the game *Four in a Row* on a circular grid. This familiarity with common benchmark angles will pay many dividends in future work with angles.

The need for more precision in angle measurement leads to theory and techniques for measuring angles by the introduction of two common measuring tools. The goniometer (goh nee AHM uh tur), or angle ruler, is a tool used in the medical field for measuring angle of motion or the flexibility in body joints, such as knees.

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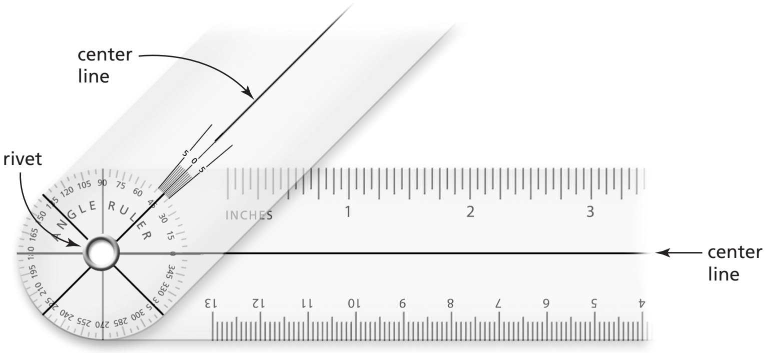
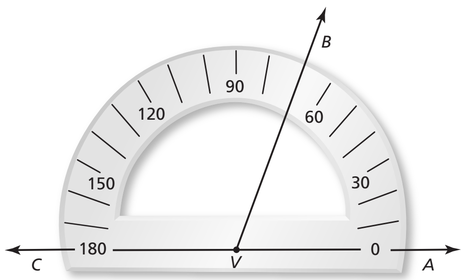
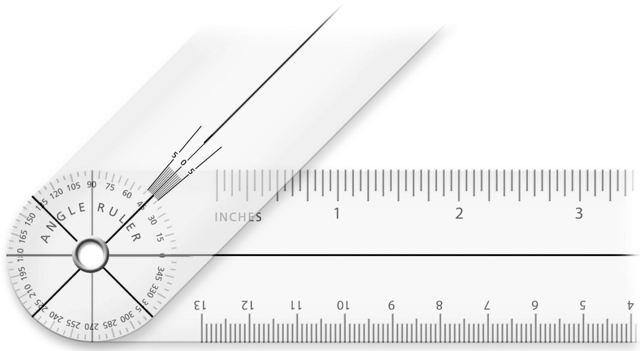


The protractor is another tool commonly used in the classroom to measure angles.



The next diagram illustrates why another method for measuring angles with the angle ruler, called the gripping method, gives the same results as placing the rivet over the vertex of the angle being measured. The overlap of the sides of the ruler forms a rhombus as you separate them.



In a rhombus, opposite angles are equal. This means that the rhombus angle at the rivet and the opposite angle are equal. The angle opposite the rivet in the rhombus is also equal to the angle between the sides, since they are vertical angles (i.e. angles formed by two intersecting lines). So, when you place a shape between the arms of the ruler, the angle at the rivet has the same measure as the angle between the arms.

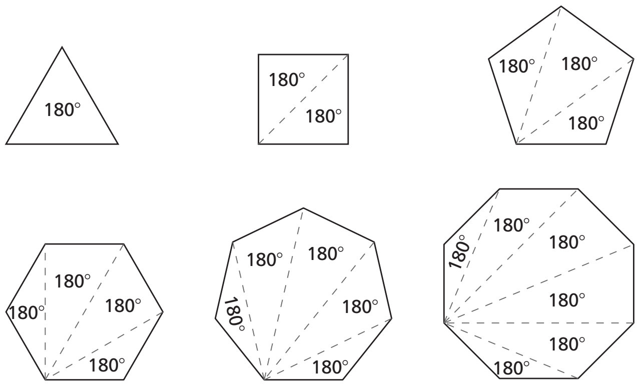
One of the critical understandings to develop about angle measurement is the fact that the measure of an angle does not depend on the lengths of the sides in a drawing. When we measure angles, we are measuring the "opening" or turn between the edges of the angle. The lengths of the two edges in a specific drawing do not affect the measure of the angle.

Angle Sums in Polygons

One of the most important theorems in all of Euclidean geometry states that the sum of the angles of any triangle is always equal to a straight angle, or 180°. This property of triangles and the application to angle sums of other polygons is developed experimentally, since, in most formal developments of geometry, its standard proof depends on a subtle axiom about parallel lines that is not developed in this Unit. Based on that property of triangles, students can then reason to more general results about the angle sum of quadrilaterals and other polygons by showing how those figures can be decomposed into triangles.

One way to reason about the angle sum in a polygon is to triangulate the polygon. Start at any vertex and draw all possible diagonals from that vertex. Triangulating a quadrilateral gives two triangles, triangulating a pentagon gives three triangles, triangulating a hexagon gives four triangles, and so on. Each time the number of sides increases by one, the number of triangles increases by one.

The number of triangles formed by drawing diagonals from a given vertex to all other nonadjacent vertices in a polygon is equal to $n - 2$.



The total interior angle measure of any polygon is $T = (n - 2) \cdot 180°$.

We can use symbols to state a rule for this pattern. If we let n represent the number of sides in a polygon, then $n - 2$ represents the number of triangles we get by triangulating the polygon. If we multiply by 180° for each triangle, we have the formula: $(n - 2) \times 180° =$ the angle sum in an n-sided polygon. Note that this is true for both regular and irregular polygons.

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Another method that students may use is to draw all the line segments from a point within a polygon to each vertex. This method subdivides the polygon into n triangles. In a quadrilateral, four triangles are formed. The number of triangles is the same as the number of vertices or sides of the quadrilateral. In the pentagon five triangles are formed. Again, the number of triangles is equal to the number of sides or vertices of the pentagon.

The sum of the angles of the four triangles in the quadrilateral is $180° \times 4$. However, this sum includes 360° around the central point. Therefore, to find the sum of the interior angles of the quadrilateral, 360° must be subtracted from the sum of the angles of the four triangles. The sum of the interior angles of the quadrilateral is $180° \times 4 - 360° = 360°$.

The sum of the angles of the five triangles formed in a pentagon is $180° \times 5$. However, this sum also includes 360° around the central point. So, to find the sum of the interior angles of a pentagon, 360° must be subtracted from the sum of the angles of the five triangles. The sum of the interior angles of the pentagon is $180° \times 5 - 360° = 540°$.

We notice that the sum of the interior angles of a quadrilateral or pentagon is 180° times the number of sides minus two. For the quadrilateral, the sum is $180° \times (4 - 2)$ and for a pentagon, the sum is $180° \times (5 - 2)$.

This method works for any polygon. For a polygon with n sides, the sum of its interior angles is: $180° \times n - 360° = 180° \times (n - 2)$.

Interior Angles of Regular Polygons

If a polygon is regular, we can find the number of degrees in one of the angles by dividing the sum by the number of angles.

The expression $\frac{(n - 2) \times 180°}{n}$ represents the measure of each angle of a regular n-sided polygon.

Students may notice that as the number of sides of a regular polygon increases, the measure for each interior angle also increases. This measure actually approaches 180°, which occurs as the shape of the polygon approaches a circle.

Exterior Angles of Regular Polygons

In a regular polygon of n sides, the sum of the interior angles is $(n - 2) \times 180°$. The measure of each angle is $\frac{(n - 2) \times 180°}{n}$.

So, the measure of each corresponding exterior angle is $180° - \frac{(n - 2) \times 180°}{n}$. The sum of n exterior angles

$$= n\left[180° - \frac{(n - 2) \times 180°}{n}\right]$$
$$= 180°n - \left(n - 2\right) \times 180°$$
$$= 180°n - 180°n + 360°$$
$$= 360°$$

This formal reasoning is probably less convincing to most Grade 7 students than the activity described in Problem 2.4 where students imagine walking around a polygon and thinking about how they complete one full turn of their direction, or a rotation of 360°.

Exploring Side Lengths of Polygons

While angles are important determinants of the shape of any polygon, side lengths play a critical role as well. Some experiments with actual polystrip pieces will make several key properties of triangles and quadrilaterals clear.

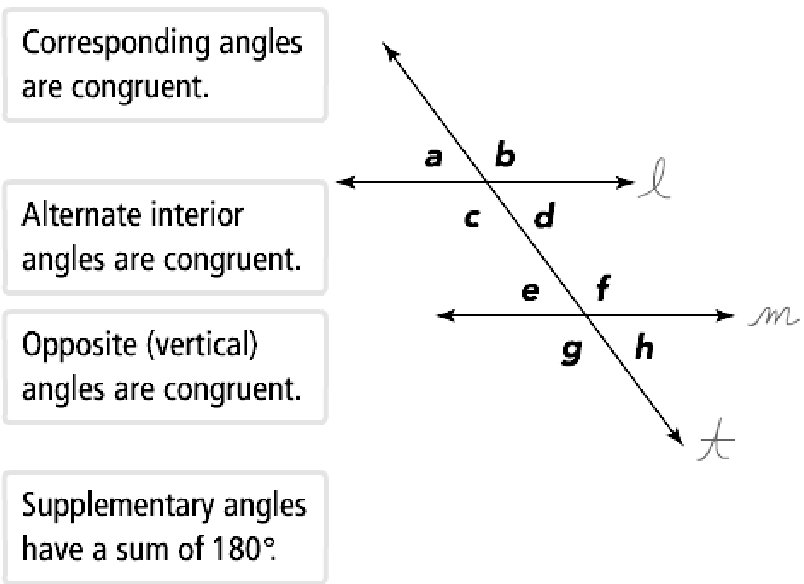
First, for any three sides to make a triangle, the sum of each pair of side lengths must be greater than the third. This side length result is called the *Triangle Inequality Theorem*.

Angles and Parallel Lines

Many important geometric structures make use of parallel lines, so it is useful to know how to check whether two given lines are parallel and how to construct parallel lines. The key principle in both tasks is the relationship between parallel lines and any third line that intersects them.

Below is a pair of parallel lines that are intersected by a third line. The line that intersects the parallel lines is called a transversal. As the transversal intersects the parallel lines, it creates several angles.

Parallel Lines and a Transversal Summary



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Angles *a* and *e*, angles *b* and *f*, angles *c* and *g*, and angles *d* and *h* are called corresponding angles. Angles *d* and *e* and angles *c* and *f* are called alternate interior angles. Parallel lines cut by a transversal create congruent (equal measure) corresponding angles and congruent alternate interior angles. Also note that if two lines intersect, they create two pairs of congruent opposite angles. In the diagram, angles *b* and *c* are congruent and so are angles *a* and *d*, *e* and *h*, and *f* and *g*. These pairs of angles are commonly called vertical angles. Angles *b* and *d* are supplementary angles. Their sum is 180°. At this point names are not stressed—only the relationship among the angles.

Parallelograms are defined in the Unit as quadrilaterals with opposite sides parallel. There are other equivalent definitions (e.g., one pair of parallel and congruent opposite sides). However, the focus on parallel lines is appropriate to the name "parallelogram."

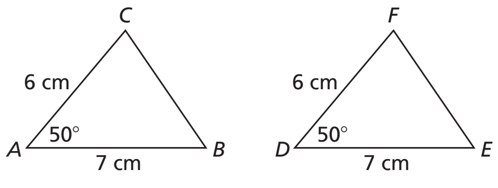
Congruence Conditions

One central theme of Investigation 3 focuses on how various combinations of side lengths and angle measurements determine the shape of a polygon. A question to pose is, "How much information about a polygon do you need to specify its shape exactly?" Clearly, you cannot replicate a triangle if all you know is the length of one side or the measure of one angle. It certainly ought to be possible to reconstruct a triangle if you know all the side lengths and angle measures, but how about two pieces of information? How about three?

One of the fundamental results of Euclidean Geometry is a set of necessary and sufficient conditions to guarantee that two triangles are congruent. The most common three such criteria are:

> If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other, the triangles will be congruent (in all parts). This condition is commonly known as the Side-Angle-Side or SAS Theorem.

If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, the triangles will be congruent (in all parts).



This condition is commonly known as the Side-Angle-Side or SAS Postulate.
In the diagram above, $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, and $\overline{AC} \cong \overline{DF}$.
So $\triangle ABC \cong \triangle DEF$ by the SAS Postulate.

If two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other, the triangles will be congruent (in all parts). This condition is commonly known as the Angle-Side-Angle or ASA Theorem.

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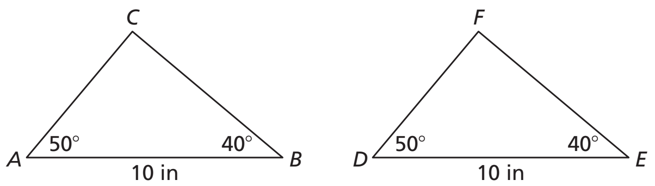

This condition is commonly known as the Angle-Side-Angle or ASA Postulate.
In the diagram above, $\angle A \cong \angle D$, $\overline{AB} = \overline{DE}$, and $\angle B \cong \angle E$.
So $\triangle ABC \cong \triangle DEF$ by the ASA Postulate.

If the three sides of one triangle are congruent to three corresponding sides of the other, the triangles will be congruent (in all parts). This condition is commonly known as the Side-Side-Side or SSS Theorem.

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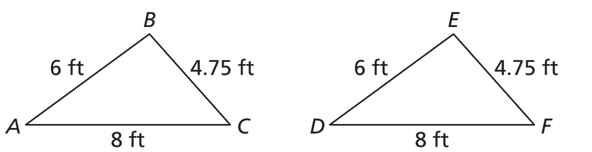

This condition is commonly known as the Side-Side-Side or SSS Postulate.
In the diagram above, $\overline{AB} = \overline{DE}$, $\overline{BC} = \overline{EF}$, and $\overline{AC} = \overline{DF}$.
So $\triangle ABC \cong \triangle DEF$ by the SSS Postulate.

There are some other special congruence criteria, but these three are the most common and useful. If you know less about the two triangles, you cannot be sure that they are congruent. You might know other sets of three measurements and still not be sure that the triangles are congruent. For example, if the three angles of one triangle are congruent respectively to the three angles of another, the triangles will be similar but not necessarily congruent. Also, there are some combinations of measurements in the SSA pattern that produce three congruent corresponding parts, but not congruent triangles.

The theme of criteria guaranteeing congruence for triangles will be revisited in much more detail in CMP Grade 8 Units. At this point what we aim for is an informal understanding of how certain kinds of knowledge about a triangle are telling. For an interesting extension of the idea, you might ask students to see how much information they would need about a quadrilateral to know its shape precisely. The simplest way to think about this question is to ask, “How could I start drawing a quadrilateral that will have the same shape as another quadrilateral?”