Mathematics Background

Extending Understanding of Rational Numbers

In Grade 6, we used integers to extend students' experience with the number line. Students were simply asked to compare positive and negative integers. Now, we approach integers in the context of rational numbers. Throughout *Accentuate the Negative*, students learn appropriate strategies for operating with rational numbers through the use of real-world problems.

Most students may be able to add, subtract, multiply, and divide positive rational numbers. However, most have not been asked to consider what the operations mean and what kinds of situations call for which operation. Students need to develop the disposition to seek ways of making sense of mathematical ideas and skills. Otherwise, they may end up with technical skills without knowing how those skills can be used to solve problems. For example, students may know how to simplify $10 \times \frac{20}{60}$, but do not understand that this expression represents the situation below.

Keith runs 10 miles per hour. How many miles does he run in 20 minutes?

One way to develop the desire to make sense of these ideas is to model such thinking in classroom conversation. Asking questions about meaning (what makes sense) as a regular, expected part of classroom discourse helps students make connections.

Sample Question

- *What operations should you use to solve the problem? How do you know?*
- *How can you write a number sentence to represent this situation?*
- *What does the number before the operation symbol represent? The number after?*
- *How do negative and positive numbers help describe the situation?*
- *Suppose you change the first number in your number sentence to be negative. What situation would the new number sentence describe?*
- *What units should the answer have? Does your number sentence support this?*
- *What model(s) for positive and negative numbers help show relationships in the problem situation?*
- *Does the order of the numbers in your expression matter?*

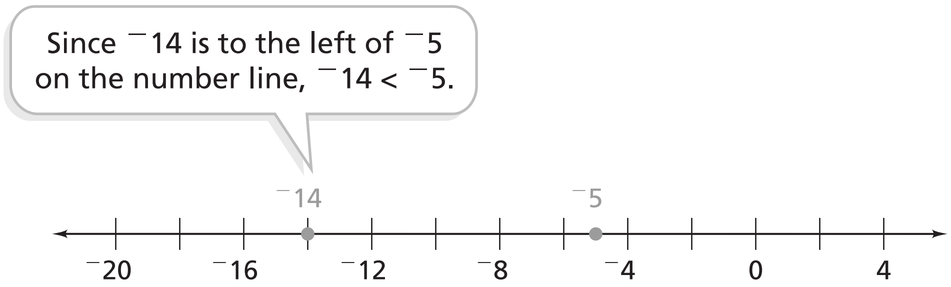
Exploring new aspects of numbers by building on and connecting to prior knowledge is likely to have two good effects. First, students will deepen their understanding of familiar numbers and operations. Second, the new numbers, negative integers and negative rational numbers, will be more deeply integrated into students' mathematical knowledge and resources.

Common Student Difficulties With Negative Numbers

Students find several things difficult about working with negative numbers.

The fact that −14 is less than −5 contradicts students' experience with positive numbers. Students need to build mental images and models in order to visualize the new comparisons and relationships between positive and negative numbers.

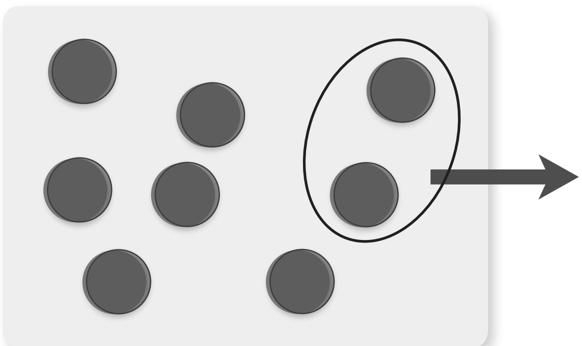
Example



Subtracting a negative number is difficult for students to understand. In this Unit, students will encounter representations and models that will help them better understand subtraction.

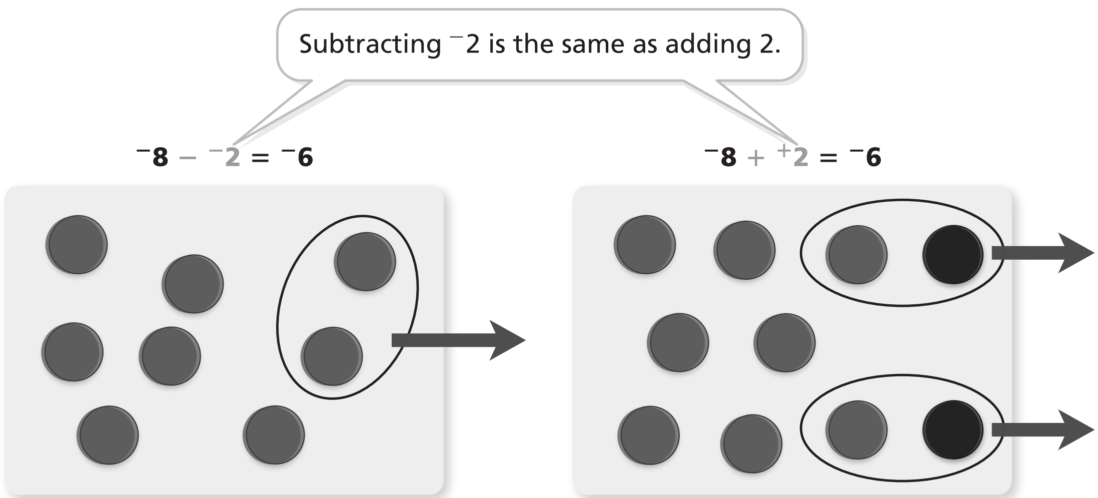
Example

⁻8 − ⁻2 = ⁻6



The understanding that subtracting a negative number is equivalent to adding the opposite of the negative number (adding a positive) must develop over time, as it is difficult for many students. Recognizing that addition and subtraction are inverse operations and that addition sentences are related to subtraction sentences helps students expand their understanding of this concept.

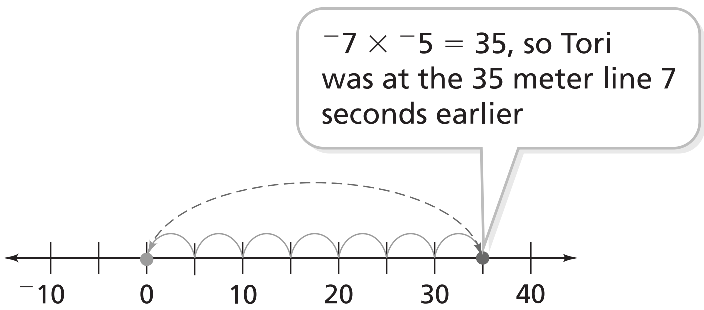
Example



The fact that multiplying two negative factors results in a positive product does not make sense to many students. In fact, the usual ways of giving meaning to multiplication, such as repeatedly adding an amount, seem of little help in making sense of expressions such as $^{-}7 \times {}^{-}5$. Providing a context for this idea helps students grasp the rule.

Example

Tori passes the 0 point running to the left at 5 meters per second. Where was she 7 seconds earlier?



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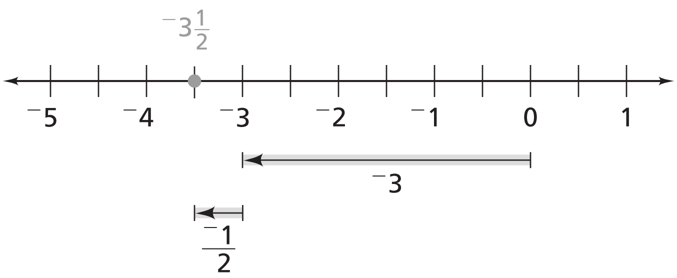
The idea that a negative rational number names a point on the number line or knowing that $-3\frac{1}{2}$ can be thought of as $-3 + {}^{-}\frac{1}{2}$ is often not transparent.

Example



Students may misread signs of rational numbers. When associated with a rational number, the + sign should be read as "positive." The − sign should be read as "negative" or "the opposite of."

Example

−12.65 is read as "negative twelve and sixty-five hundredths," not "minus twelve point six five."

+16 is read as "positive sixteen," not "plus sixteen."

In this Unit, students use integer chip and number line models to approach these difficult concepts related to negative numbers. Students use fact families and relationships between operations to further solidify their understanding.

Inequalities

Inequalities are introduced in the Grade 6 Unit *Comparing Bits & Pieces*. We use integers as a way to name points to the left of 0 on the number line. Students then use the number line to compare integers.

Example

For each pair of numbers, decide if the first is less than, equal to, or greater than the second number.

$^{-}2$ or $^{-}7$

14 or $^{-}22$

8 or 12

In *Variables and Patterns*, students solve equations associated with real-world problems. They then incorporate inequalities into those same situations to solve related questions. Students then represent these solutions on a number line.

Example

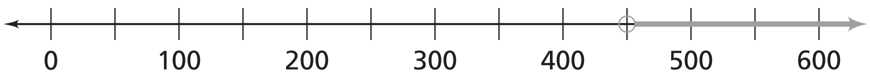
- If tickets sell for \$21 each, how many tickets n must be sold to make an income I of \$9,450?

  Use the equation $21n = I$.

- How many tickets must be sold to have more than \$9,450 income?

  Use the inequality $21n > 9, 450$or$9 , 450 < 21n$.

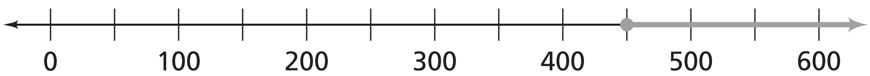
$21n > 9.450$



- How many tickets must be sold to have at least \$9,450 of income?

  Use the inequality $21n \geq 9, 450$ or $9, 450 \leq 21n$.

$21n \geq 9.450$

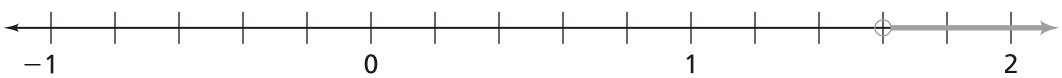


Be sure that students use the inequality symbols as opportunities to compare two quantities. They need to be able to interpret an inequality in two related ways. For example, $5 < 7$ and $7 > 5$ *both* mean that 5 is less than 7. Both inequalities also mean that 7 is greater than 5. This is particularly helpful in interpreting an inequality in which the variable is on the right side, such as $5 > n$.

In *Accentuate the Negative*, students extend their understanding of inequalities to include all rational numbers, with particular focus on comparing values less than zero. In this Unit, simple inequalities are enhanced with contexts that naturally use integers such as temperatures and money gained/money spent. You can represent such situations on the number line. Understanding how to compare rational numbers will help students develop algorithms for operations with rational numbers and then make sense of the answers that result from these operations.

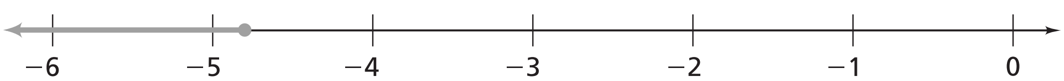
An inequality can be graphed on a number line. An open circle is used when graphing an inequality with $<$ or $>$, because the number is not part of the solution. The solution set approaches the exact point to which the variable is being compared, but does not include that point.

$p > 1\frac{3}{5}$



A closed circle is used for $\leq$ or $\geq$ because the number is part of the solution. The solution set includes the exact point to which the variable is being compared as well as all rational and irrational numbers leading up to it.

$g \leq -4.8$



Models for Integers and the Operations of Addition and Subtraction

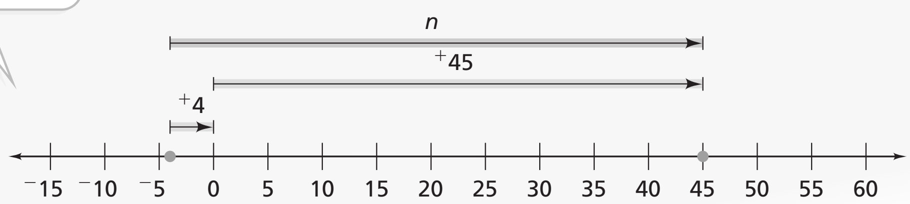
Number Line

The number line is a model that is used throughout the number system units. It was first introduced in *Comparing Bits and Pieces* to develop understanding of equivalent fractions and decimals. It was used in *Let's Be Rational* and *Decimal Ops* to help students compute expressions containing fractions and decimals. It is used later in *Looking for Pythagoras* to introduce square roots and irrational numbers.

In this Unit, students use the directed distance model with the number line to visualize adding and subtracting integers. Here is a situation that students encounter that uses both *distance* and *direction* as ways to consider integers.

On a number line, this change can be shown using an arrow.

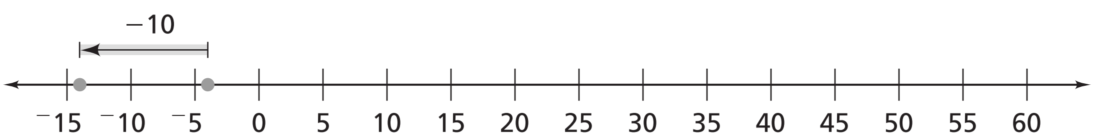
The world record for fastest rise in outside air temperature occurred in Spearfish, South Dakota, on January 22, 1943. The temperature rose from $^{-}4$°F to 45°F in two minutes. What was the change in temperature over those two minutes?



1.3 From Sauna to Snowbank: Using a Number Line

The change in temperature is $^{+}49$ °F. The sign of the change in temperature shows the *direction* of the change. Since the change is positive, the temperature increased, or moved to the right on the number line. The *distance* of the change is the difference of the start and end points, or 49 in this case.

If the temperature had dropped 10 °F, students would write the change as $^{-}10$ °F to show the size and direction of the change. The change is negative, so the temperature decreased, or moved to the left on the number line. The distance, or size, of the change is 10.



$^{-}4^\circ + \quad n \quad 1 = {}^{-}14^\circ$

$^{-}4^\circ + {}^{-}10^\circ = {}^{-}14^\circ$

Students can write these equations without the degree markers as long as they remember what the answer means.

To facilitate the development of the algorithms, the absolute value concept is introduced in Investigation 2 as a way to talk about distance on the number line. It is the value of the difference between two numbers when direction is not considered.

Ensure that students do not confuse the concept of **absolute value** with opposites. They may think that since $|{}^{-}6| = 6$, then $|6| = {}^{-}6$. Students must understand that absolute value measures distance, not direction, and that distance is always positive.

Chip Model

Colored chips can also be used to develop a strategy for adding and subtracting integers. Using this model requires an understanding of opposites. For example, ${}^{-}3$ and ${}^{+}3$ are opposite because ${}^{+}3 + {}^{-}3 = 0$. The numbers are equidistant from zero on the number line.

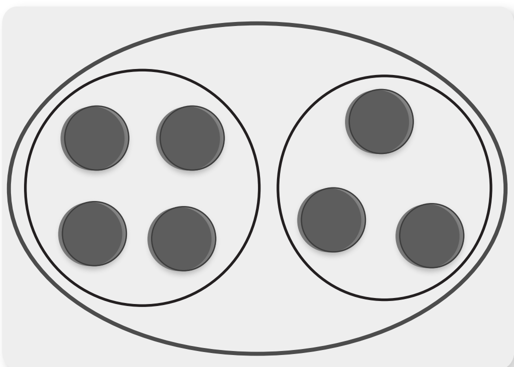
The chip model uses one color of chips (black) to represent positive integers and another color (red) to represent negative integers. (**Note:** You may use any collection of two-color chips—designate a color to be positive and a color to be negative.) Because each chip represents 1 unit, either positive or negative, a red chip and a black chip are thought of as opposites. Combining two opposite chips makes zero (${}^{+}1 + {}^{-}1 = 0$).

To use the model with addition, begin with an empty chip board. Place chips on the board to represent each addend. If the integer is positive, place that number of black chips on the board. If the integer is negative, place that number of red chips on the board.

Same Signs

If the two integers being added have the **same sign**, the sum is the total number of chips on the board. For example, to add ${}^{-}4 + {}^{-}3$, place 4 red chips on the board and then another 3 red chips on the board for a total of 7 red chips (representing a sum of -7).

$${}^{-}4 + {}^{-}3 = {}^{-}7$$



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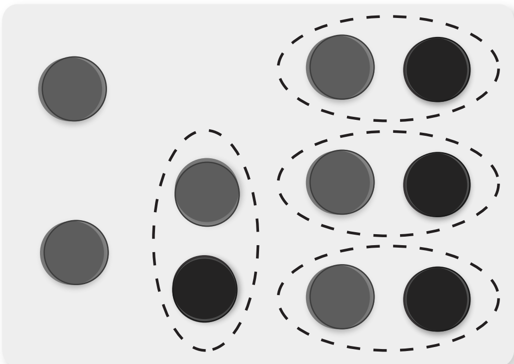
Different Signs

If the integers being added have **different signs**, place the appropriate number of red and black chips on the board to represent each addend. Simplify the board by removing red-black (opposite) pairs of chips. The value on the board does not change since the red-black pairs have a sum of zero. The chips that remain unmatched represent the sum of the two integers. Consider this problem:

> *Tate owes his sister $6 for helping him cut the lawn. He earns $4 delivering papers. Is Tate "in the red" or "in the black"?*
>
> 1.4 In the Chips

Using a collection of 4 black chips and 6 red chips on a chip board, you can represent the combination of expense and income. The net worth, or total value, is "in the red" two dollars, or -2 dollars. This problem may be represented with the number sentence, $^{-}6 + {}^{+}4 = {}^{-}2$.

$$\mathbf{^{-}6 + {}^{+}4 = {}^{-}2}$$

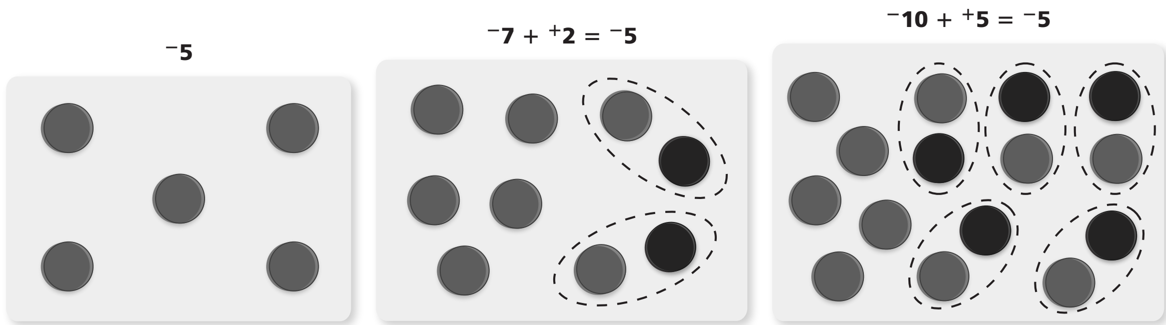


Numerically, you can rewrite -6 as $^{-}2 + {}^{-}4$ so that the -4 can be paired with the $+4$ to make zero:

$$\begin{aligned} ^{-}6 + {}^{+}4 &= {}^{-}2 + {}^{-}4 + {}^{+}4 \\ &= {}^{-}2 + 0 \\ &= {}^{-}2 \end{aligned}$$

After the paired chips are removed, 2 red chips remain.

In the chip model, integers may be represented using different combinations of chips. For example, -5 can be shown with 5 red chips ($^{-}5 = {}^{-}5$), with 7 red chips and 2 black chips ($^{-}5 = {}^{-}7 + {}^{+}2$), or with 10 red chips and 5 black chips ($^{-}5 = {}^{-}10 + {}^{+}5$).



This flexibility in representing integers with different combinations of positive and negative chips helps students model subtraction. Subtraction involves representing a quantity with chips and then removing ("taking away") the number of chips necessary.

Representing Integers with Combinations of Chips

Problem	Show	Remove	Answer
$^{+}7 - {}^{+}5$	7 black	5 black	$^{+}7 - {}^{+}5 = {}^{+}2$
$^{-}8 - {}^{-}3$	8 red	3 red	$^{-}8 - {}^{-}3 = {}^{-}5$
$^{+}7 - {}^{-}2$	9 black and 2 red	2 red	$^{+}7 - {}^{-}2 = ({}^{+}9 + {}^{-}2) - {}^{-}2 = {}^{+}9$
$^{-}5 - {}^{-}7$	7 red and 2 black	7 red	$^{-}5 - {}^{-}7 = ({}^{+}2 + {}^{-}7) - {}^{-}7 = {}^{+}2$
$^{-}4 - {}^{+}2$	6 red and 2 black	2 black	$^{-}4 - {}^{+}2 = ({}^{-}6 + {}^{+}2) - {}^{+}2 = {}^{-}6$
$^{+}3 - {}^{+}7$	7 black and 4 red	7 black	$^{+}3 - {}^{+}7 = ({}^{-}4 + {}^{+}7) - {}^{+}7 = {}^{-}4$

The last four problems require representing the minus end as a combination of red and black chips.

Jeremy earns $10 mowing a lawn. He used his credit card to rent the lawn mower. Jeremy now owes his credit card company $15. How much money does Jeremy have?

This problem may be modeled using chips by representing the $10 earned with a combination of 15 black chips and 5 red chips ($10 = 15 + {}^{-}5$). With this alternative representation of 15, $15 or 15 black chips can be "taken away." Five red chips are left to represent the $5 that Jeremy is "short." Two different number sentences are applicable:

$10 + {}^{-}15 = {}^{-}5$ and $10 - {}^{+}15 = {}^{-}5$

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As students use models to add and subtract integers, they notice that it may be helpful to restate an addition problem as a subtraction problem or vice versa. This involves using opposites of numbers.

Example

Adding $^{-}8$ is the same as subtracting $^{+}8$.

$^{+}12 + {}^{-}8 = {}^{+}12 - {}^{+}8$

Subtracting $^{-}7$ is the same as adding $^{+}7$.

$^{+}5 - {}^{-}7 = {}^{+}5 + {}^{+}7$

Students can use the generalizations made about integers to extend their work to negative and positive rational numbers.

Fact Families

Fact families are used in this Unit to help students understand the relationship between addition and subtraction and between multiplication and division. Fact families were introduced in the number sense units in Grade 6.

Fact Families

Example 1
$^{-}7 + {}^{+}2 = {}^{-}5$
$^{+}2 + {}^{-}7 = {}^{-}5$
$^{-}5 - {}^{+}2 = {}^{-}7$
$^{-}5 - {}^{-}7 = {}^{+}2$

Example 2
$^{-}12 \div {}^{-}6 = {}^{+}2$
$^{-}12 \div {}^{+}2 = {}^{-}6$
$^{-}6 \times {}^{+}2 = {}^{-}12$
$^{+}2 \times {}^{-}6 = {}^{-}12$

Fact families are also used to find missing values. By rewriting an equation in one of the other forms in the fact family, students can isolate variables, which can make solutions more accessible.

Equations in a Fact Family

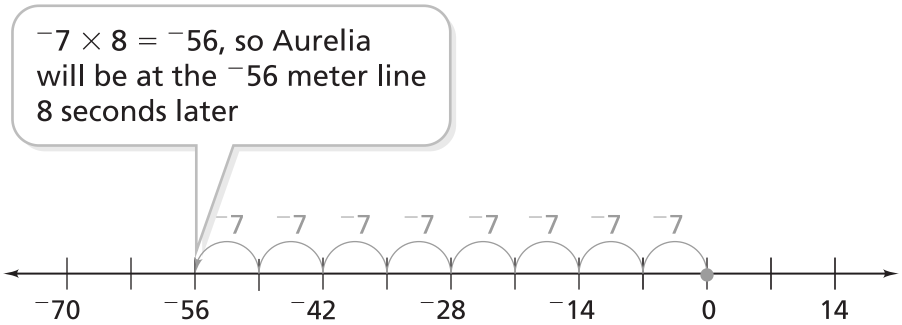
Equation	Related Equation	Solution
$^{+}4 + n = {}^{+}43$	$^{+}43 - {}^{+}4 = n$	$n = {}^{+}39$
$^{-}6n = 42$	$42 \div {}^{-}6 = n$	$n = {}^{-}7$

Models and the Operations of Multiplication and Division

Number Line

Multiplication can be explored by counting occurrences of fixed-size movement along a number line. The direction of the movement introduces negative and positive signs.

Aurelia passes the 0 point running to the left at 7 meters per second. Where is she 8 seconds later?



Since Aurelia is running to the left, her speed has a negative direction (-7). In 8 seconds, Aurelia runs $^{-}7 \times {}^{+}8$, or -56 meters. So, Aurelia will be at the -56-meter line.

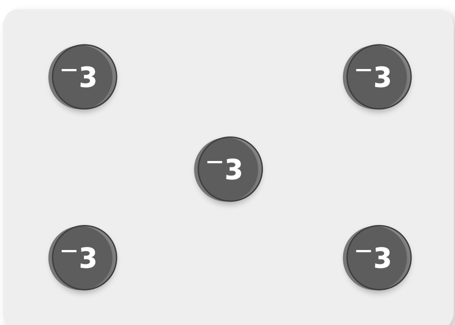
Tori passes the 0 point running 4 meters per second to the left. Where was she 6 seconds earlier?

Since Tori is running to the left, her speed has a negative direction (-4). In 6 seconds, Tori runs $^{-}4 \times 6$, or -24 meters. To find where Tori was 6 seconds *earlier*, find the opposite of -24 meters, which is $+24$ meters. So, 6 seconds earlier, Tori was at the $^{-}4 \times {}^{-}6$, or $+24$-meter line. Visit *Teacher Place* at mathdashboard.com/cmp3 to see the complete video.

Chip Model

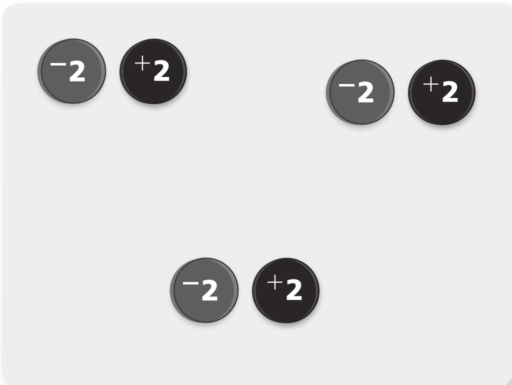
Multiplication can also be represented by using the chip model, provided that one of the factors is positive. Each chip can represent the same positive or negative integer. The number of chips the student places on the board represents the other factor (this must be positive). Students can then add the values of all the chips, or students can multiply the value of each chip by the number of chips.

$5 \times {}^{-}3 = {}^{-}15$



Multiplying two negative integers is difficult to model on a chip board and is generally considered too difficult for Grade 7 students. However, the image below does provide a way in which the model can be used.

In the image above, five groups of -3 are **added** to the chip board, modeling the expression $5 \times {}^{-}3$. The following image shows how you can use the chip model to find the product ${}^{-}3 \times {}^{-}2$. In this case, you must **take away** three groups of -2.



Place three zero pairs of chips on the board, which does not change the value of the board. (In this case, a zero pair is a black chip labeled $+2$ and a red chip labeled -2.) This allows you to take away three groups of -2. The end result left on the board is $+6$, so ${}^{-}3 \times {}^{-}2 = {}^{+}6$.

Relating Division to Multiplication

Relating division to multiplication helps develop division with integers. A multiplication fact can be used as the basis for creating two related division facts. By developing division in this way, students can determine the sign of the answer to a division problem. Then, students can generalize rules for determining the sign of a quotient in a division problem.

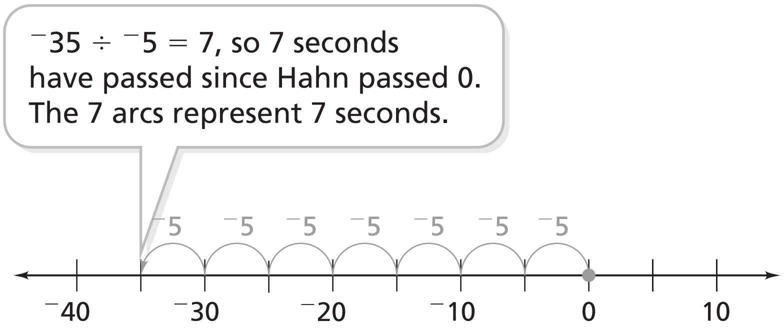
Related Multiplication and Division Equations

Equation	Related Equation	Solution
${}^{-}10 \div {}^{-}2 = n$	${}^{-}2 \times n = {}^{-}10$	$n = {}^{+}5$
${}^{+}24 \div {}^{-}8 = n$	${}^{-}8 \times n = {}^{+}24$	$n = {}^{-}3$

Division With a Number Line

Students can also use the number line as a model for division. Using the same running situation as multiplication, students can divide a distance into fixed-size occurrences.

Hahn is 35 meters left of the 0 point. If he were running 5 meters per second to the left, how many seconds have passed since Hahn was at the 0 line?



Each arc represents the -5 meters that Hahn runs each second. There are 7 arcs from 0 to -35. $^{-}35 \div {}^{-}5 = {}^{+}7$, so 7 seconds have passed since Hahn was at the 0 point.

Some Notes on Notation

Writing integers with raised signs helps students avoid confusion with symbols for addition and subtraction. However, most software and writing in mathematics do not use the raised signs. Students use raised signs for the first two Investigations, after which they use the standard notation.

- Positive numbers are usually written without a sign.
  - $^{+}3 = 3$
  - $^{+}7.5 = 7.5$
- Negative numbers are usually written with a dash like a subtraction sign.
  - $^{-}3 = -3$
  - $^{-}7.5 = -7.5$
- Parentheses can help by visually separating the signs of integers and operation symbols.
  - $^{-}5 - {}^{-}8 = -5-(-8)$
- The subtraction symbol also indicates the opposite of a number.
  - -8 represents the opposite of 8.
  - $-(-8)$ represents the opposite of -8, or 8.
  - $-h$ represents the opposite of h.
  - $-(-h)$ represents the opposite of $-h$, or h.
- The times symbol, multiplication dot, or parentheses can be used to indicate multiplication. If one or both factors of a product are variables, no symbol is needed.
  - $3 \times 5 = 3 \cdot 5$
  - $3 \cdot (4 + 5) = 3 \times (4 + 5) = 3(4 + 5)$
  - $6 \times g = 6g$

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Rational Numbers

In this Unit, students use integers to find patterns for adding, subtracting, multiplying, and dividing integers. Students use the rules they discovered for integers to compute with rational numbers. These rules and patterns work in either case. After becoming familiar with positive rational number operations in Grade 6, students can focus on using operations with negative rational numbers in Grade 7. In Grade 8, students will study irrational numbers. The rational and irrational numbers make up the set of real numbers.

Order of Operations

Order of operations rules are reinforced in this Unit. The rules have been used in prior grades, but we revisit the Order of Operations with a special focus on negative numbers.

1. Compute any expressions within parentheses.

Examples

$$(-7 - 2) + 1 = -9 + 1$$
$$= 8$$
$$(1 + 2) \times (-4) = 3 \times (-4)$$
$$= -12$$

2. Compute any exponents.

Examples

$$-2 + 3^2 = -2 + 9$$
$$= 7$$
$$6 - (-1 + 4)^2 = 6 - 3^2$$
$$= 6 - 9$$
$$= -3$$

3. Multiply and divide in order, from left to right.

Examples

Multiply first. $1 + 2.8 \times 4.01 = 1 + 11.228$

$= 12.228$

Divide first. $200 \div 10 \times \frac{1}{2} = 20 \times \frac{1}{2}$

Multiply second. $= 10$

4. Add and subtract in order, from left to right.

Examples

$1 - 2 = 3 \times 4 = 1 - 2 + 12$

Subtract first. $= -1 + 12$

Add second. $= 11$

Properties of Rational Numbers

The Commutative Property of Addition and Multiplication is reinforced in this Unit. Students find that this property holds for multiplication and addition of rational numbers, but does not hold for subtraction or division of rational numbers.

The Distributive Property of Multiplication over Addition or Subtraction is also discussed and modeled. Rectangles similar to those introduced in *Prime Time* are used to demonstrate the Distributive Property. Students apply this property when solving word problems and computing expression.

The Identity Property and the Inverse Property of Addition and Multiplication are introduced in this Unit.The Associative Property is explored in an ACE Exercise. These properties are revisited in several succeeding units, particularly the algebra units.

The properties that students will encounter in this Unit are listed below.

Properties of Rational Numbers

Property	Algebra	Example
Commutative Property of Addition Changing the order of addends does not change the sum.	$a + b = b + a$	$-8 + 7 = 7 + (-8)$
Commutative Property of Multiplication Changing the order of factors does not change the product.	$a \times b = b \times a$	$-16 \times \left(-\frac{2}{3}\right) = -\frac{2}{3} \times \left(-16\right)$
Distributive Property The product of a number and a sum (or difference) can be rewritten as the sum (or difference) of two products.	$a(b + c) = ab + ac$ $a(b - c) = ab - ac$	$4(7.08 + 6.4) = 4(7.08) + 4(6.4)$ $13(-8 - 2) = 13(-8) - 13(2)$
Identity Property of Addition Adding 0 to a number does not change its value.	$a + 0 = a$	$-17 + 0 = -17$
Inverse Property of Addition The sum of a number and its opposite is 0.	$a + (-a) = 0$	$9 + (-9) = 0$
Identity Property of Multiplication Multiplying a number by 1 does not change its value.	$a \times 1 = a$	$-5.46 \times 1 = -5.46$
Inverse Property of Multiplication The Product of a nonzero number and its reciprocal is 1.	$\frac{a}{b} \times \frac{b}{a} = 1$, where $a, b \neq 0$	$\frac{3}{5} \times \frac{5}{3} = 1$

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