Mathematics Background

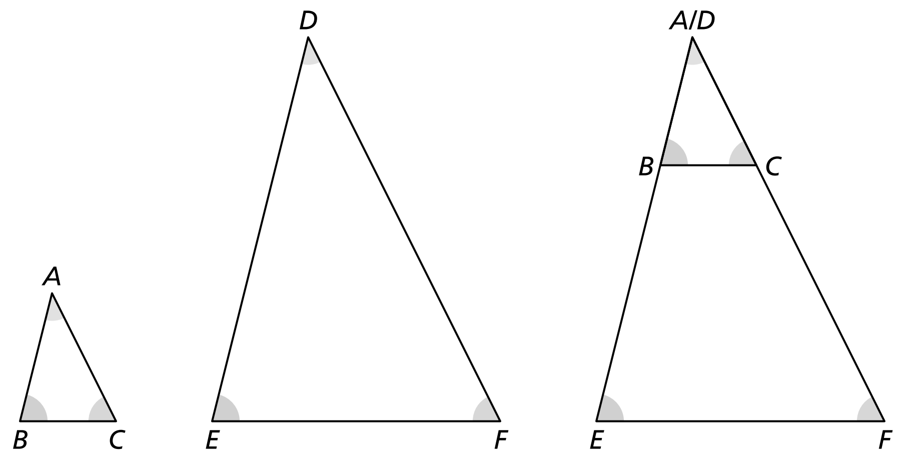
Introduction to Similarity

Students may initially have difficulty with the concept of similarity because of the way the word is used in everyday language. Family members are "similar" if their behaviors are somewhat alike. Houses can be "similar" if they have certain shared attributes, such as the same number of bedrooms or the same outside color. Students begin this Unit by informally exploring what it means for two geometric figures to be *similar*.

It is generally recognized that understanding proportional reasoning is an important stage in cognitive development. Students in the middle grades often experience difficulty with ideas of scale. They confuse *adding* situations with *multiplying* situations. Situations requiring comparison by addition or subtraction come first in students' experience with mathematics and often dominate their thinking about any comparison situation, even those in which scale is the fundamental issue. For example, when considering the dimensions of a rectangle that began as 3 units by 5 units and was enlarged to a similar rectangle with short side of 6 units, many students will say the long side is now 8 units rather than 10 units. They add 3 units to the 5 units rather than multiply the 5 units by 2, the scale factor. These students may struggle to build a useful conception that will help them distinguish between situations that are additive and those that are multiplicative (calling for scaling up or down).

The activities at the beginning of *Stretching and Shrinking* build on students' notions about similarity as they explore figures with the same shape. They draw similar figures using rubber bands and coordinate plane rules. Enlarging or shrinking pictures with a photocopier provides another familiar context for exploring similar figures. See the video below for a demonstration of how to enlarge a figure with a rubber-band stretcher. You can show this video to your students before they begin Problem 1.1 if you think they would benefit from the demonstration. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.

Early on, students notice that some attributes of similar figures are the same, while others are not. For example, corresponding angle measures of similar figures are the same, but corresponding side lengths are different. The differences in corresponding side lengths, however, are predictable.



Through the activities in *Stretching and Shrinking*, students will grow to understand that the everyday use of the word *similar* and its mathematical use may be different. For students to determine definitively whether two figures are similar, similarity must have a precise mathematical definition.

Defining Mathematical Similarity

Two figures are similar if

- the measures of their corresponding angles are equal;
- the lengths of their corresponding sides increase by the same factor, called the scale factor.

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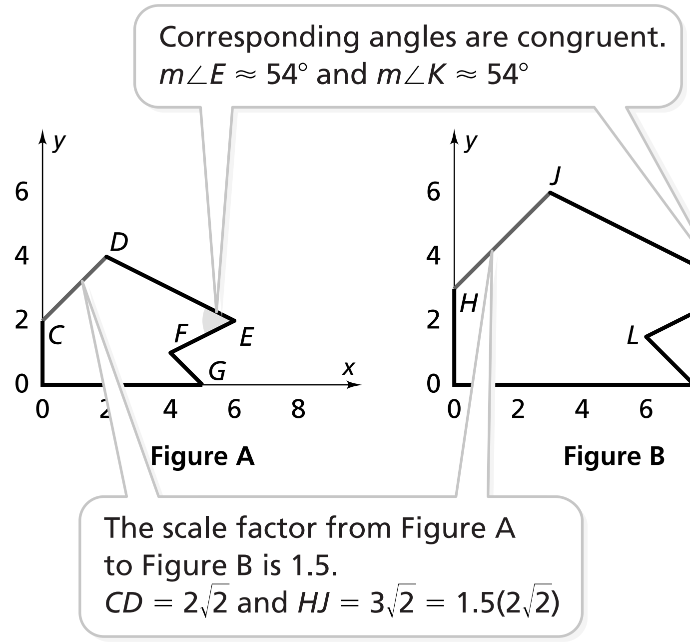
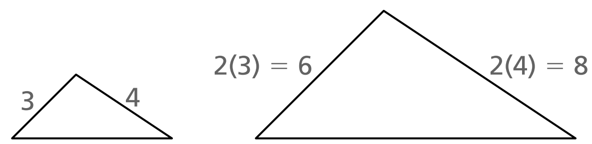
Figures A and B below are similar.



The corresponding angle measures of Figure A and Figure B are equal. Each side length in Figure A can be multiplied by 1.5 to produce the corresponding side length in Figure B. Thus, the scale factor from Figure A to Figure B is 1.5. Figure A stretches, or is enlarged, to become Figure B. You can also say that the scale factor from Figure B to Figure A is $\frac{1}{1.5}$, or $\frac{2}{3}$. Figure B shrinks, or is reduced, to become Figure A.

Throughout this Unit, the scale factor can be thought of as the unit of proportionality. For example, if the scale factor from one triangle to another similar triangle is 2, we can say that for every unit of length in the side lengths of one triangle, there are 2 units of length in the corresponding side lengths of a similar triangle. If *a* and *b* are two side lengths of the original triangle, then 2*a* and 2*b* are the corresponding side lengths of the other triangle. This is true for all similar figures. The scale factor is used to identify similar figures and also to find missing measurements.

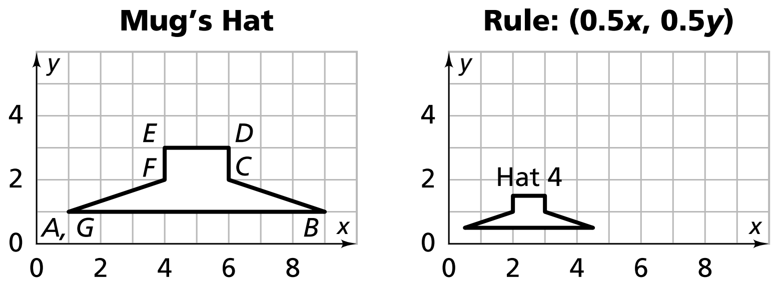


Making Similar Figures or Scale Drawings

The rubber-band stretcher introduced in Problem 1.1 is a tool for producing a scale drawing. It does not give precise results, but it is an effective way to introduce students to similar figures. The algebraic rules introduced in Problem 2.1, which specify how coordinates change, produce more precise transformations.

Students draw figures on a coordinate system and use algebraic rules to transform them into similar figures. For example, if the coordinates of a figure are multiplied by 4, the algebraic transformation is from (x, y) to $(4x, 4y)$. In general, if the coordinates of a figure are (x, y), algebraic rules of the form $(nx + a, ny + b)$ will transform the figure into a similar figure with a scale factor of n. These algebraic rules are called *similarity transformations*.

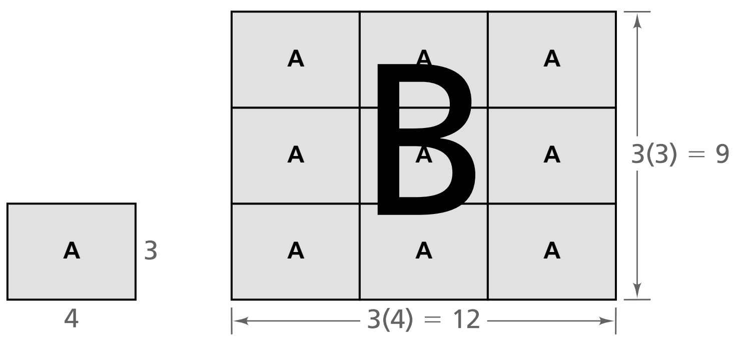
In the figures below, Mug's Hat is transformed into Hat 4 by the algebraic rule $(0.5x, 0.5y)$.



The word *transformation* is not introduced in this Unit, but it will be explored and expanded upon in Grade 8.

Relationship of Area and Perimeter in Similar Figures

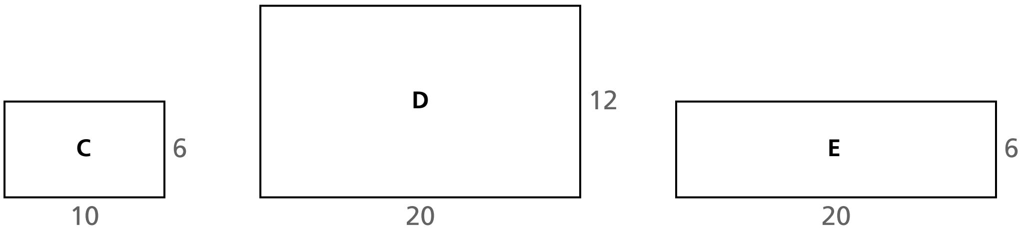
The perimeters of Rectangle A and Rectangle B below are related by a scale factor of 3.



The area increases by the square of the scale factor, 3^2, or 9. You can divide Rectangle B into nine rectangles that are each congruent to Rectangle A.

Similarity of Rectangles

Since all of the angles in rectangles are right angles, you only need to check the scale factors relating the lengths of corresponding sides. For example, Rectangles C and D below are similar, but neither is similar to Rectangle E.



The scale factor from Rectangle C to Rectangle D is 2, because twice the length of each side of Rectangle C gives the length of the corresponding side of Rectangle D. The scale factor from Rectangle D to Rectangle C is $\frac{1}{2}$, because the length of each side of Rectangle D multiplied by $\frac{1}{2}$ gives the length of the corresponding side of Rectangle C.

Rectangle E is not similar to Rectangle C, because the lengths of corresponding sides do not increase by the same factor. The longer sides of Rectangles C and E have a scale factor of 2, while the shorter sides have a scale factor of 1.

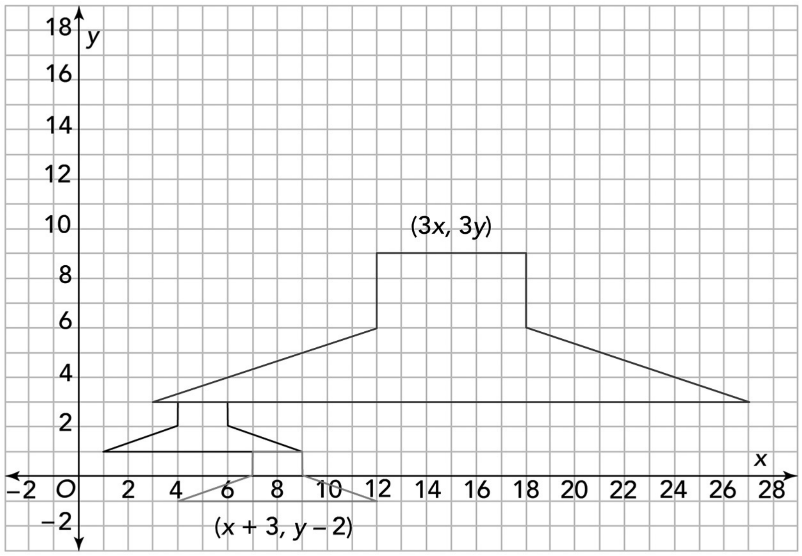
Note: The scale factor (2) from Rectangle C to Rectangle D is the reciprocal of the scale factor ($\frac{1}{2}$) from Rectangle D to Rectangle C. This relationship is always true of similar figures. The scale factor from the larger figure to the smaller figure is the reciprocal of the scale factor from the smaller figure to the larger figure.

Similarity Transformations and Congruence

In general, algebraic rules of the form (nx, ny) are called similarity transformations, because they will transform a figure in the plane into a similar figure in the plane.

For two similar figures described by the algebraic rules (dx, dy) and (nx, ny), the scale factor is the ratio of the coefficients, $\frac{n}{d}$. The figure described by the rule (x, y) is a special case in which the coefficient of both x and y is 1. If this figure is compared to the figure described by the rule (nx, ny), then $\frac{n}{1}$, or n, is the scale factor from the original figure to the image. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete animation.

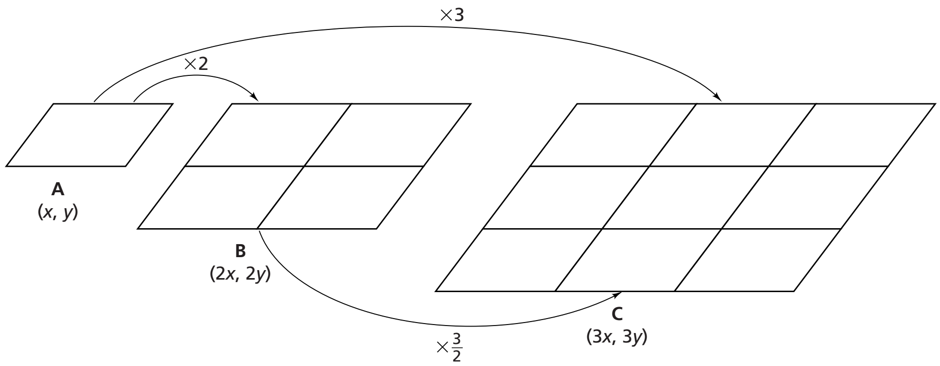
Enlargements and reductions form similar figures.
Translations form congruent figures.



In the diagram below, the scale factor from Figure A (x, y) to Figure B $(2x, 2y)$ is $\frac{2}{1}$, or 2. The scale factor from Figure A (x, y) to Figure C $(3x, 3y)$ is $\frac{3}{1}$, or 3.

Figures B $(2x, 2y)$ and C $(3x, 3y)$ are also similar. As mentioned before, the scale factor is the ratio of the coefficients: $\frac{3}{2}$.

Note that similarity is transitive. If Figure A is similar to Figure B and Figure B is similar to Figure C, then Figure A is similar to Figure C.



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In Problem 2.2, students will see that adding to the x- and/or y-coordinate moves the figure around on the grid but does not affect its size. This means that a more general form of similarity transformation is $(nx + a, ny + b)$. Rules of this form, in which the coefficient of both x and y is 1, such as $(x + 3, y - 2)$, move the figure around but do not change the shape or size of the figure. The image in such a transformation, therefore, is congruent to the original figure.

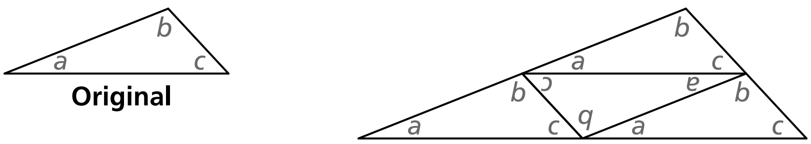
Congruent is a term from *Shapes and Designs*. Note that the scale factor between two congruent figures is 1. Therefore, congruent figures are also similar.

There are other transformations in the plane that preserve congruence, such as flips and turns. These are studied in the Grade 8 Unit *Butterflies, Pinwheels, and Wallpaper*.

Comparing Area in Two Similar Figures Using Rep-Tiles

In the first two Investigations, area is explored informally. In Investigation 3, students use rep-tiles to explore the relationship between area and scale factor. **Rep-tiles** are figures that are made from congruent copies of a shape such that the new figure is similar to the original shape.

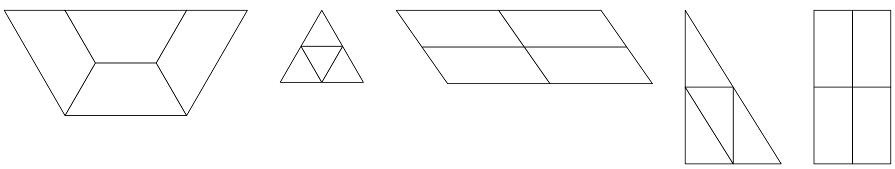
Students find that when they apply a scale factor of 2, they need four copies of the original figure. It is generally surprising to students that if you apply a scale factor of 2 to a figure, the area becomes 4 times as large. One approach is to have students calculate the area of a figure and that of its image and compare the results. In the case of rep-tiles, students are really measuring area using the original figure as the unit, rather than using square inches or square centimeters. For example, the image below is made up of four copies of the original triangle:



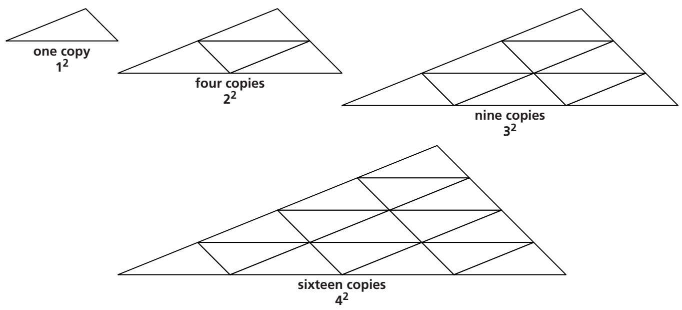
Original **four copies**

From the diagram above it is fairly easy to see that corresponding angles have equal measures.

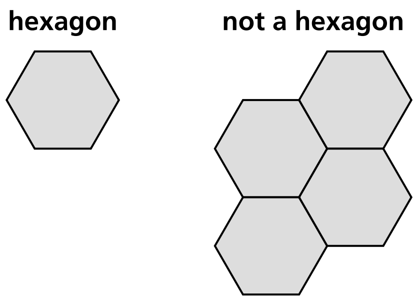
The following examples are also rep-tiles with a scale factor of 2 from the smaller shape to the larger shape. Visit Teacher Place at mathdashboard.com/cmp3 to see the image gallery.



A rep-tile that increases the figure by a scale factor of n increases the area of the figure by n^2. The graphic below illustrates this pattern:



Sometimes students incorrectly generalize ideas about rep-tiles; they may erroneously think that any figure that tiles is a rep-tile. Figures that tile (such as hexagons) may not make a larger, similar figure:



In addition, any figure can be transformed into a larger or smaller image, regardless of whether the figure can tile the plane. Rep-tiles are special because they make area comparisons easy.

Equivalent Ratios

In similar figures, there are several types of equivalent ratios. Some are formed by comparing lengths within a figure. Others are formed by comparing lengths between two figures. For the rectangles below, the ratio of length to width is $\frac{10}{6}$, or $1.\overline{6}$, for Rectangle P; and $\frac{20}{12}$, or $1.\overline{6}$, for Rectangle R. This ratio describes the general shape of the similar rectangles.

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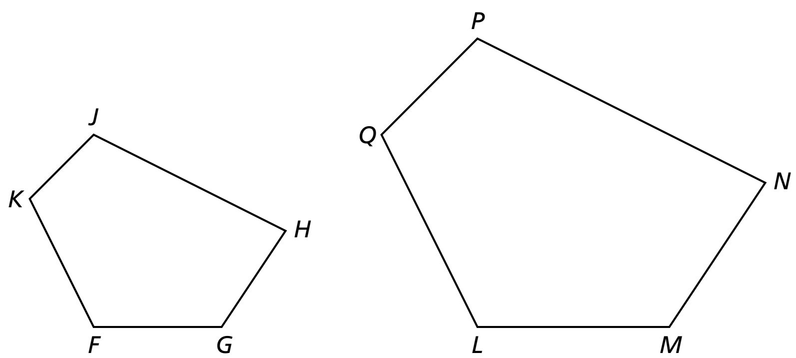
You can also look at the ratios of corresponding sides across two figures by comparing the width of P to the width of R and the length of P to the length of R. These ratios are $\frac{12}{6}$ and $\frac{20}{10}$, respectively. These ratios are equivalent, and they are also equivalent to 2, the scale factor. This type of ratio is not formally discussed in the Unit, except in ACE Exercise 42 of Investigation 4. Students use it informally throughout the Unit, however, when they divide corresponding side lengths of two similar figures to find the scale factor.

These ratios can be found for any two polygons with congruent corresponding angles. For two polygons with equal corresponding angle measures, you can test for similarity either by checking the ratios of sides within each figure or by checking the scale factor between corresponding sides. Given the two figures below, if $\frac{FG}{GH} = \frac{LM}{MN}$, $\frac{GH}{HJ} = \frac{MN}{NP}$, and $\frac{HJ}{JK} = \frac{NP}{PQ}$, then the figures are similar.



Similarity of Triangles

For most polygons, when considering similarity, you must make sure that the lengths of corresponding sides increase by the same scale factor and that corresponding angle measures are equal. For triangles, you need to check only one of these characteristics to determine if two triangles are similar. These facts about triangles are only hinted at in the Unit, however. At this stage of their development, students should use the general definition of similarity that applies to all polygons:

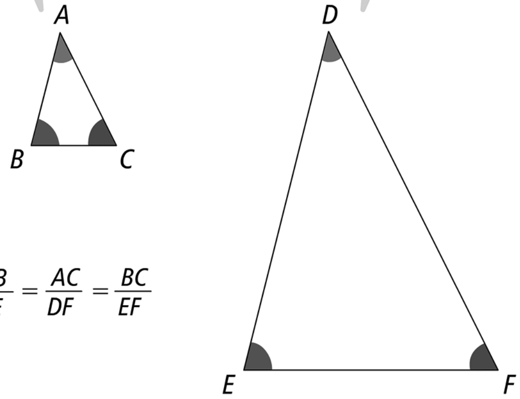
Corresponding angle measures are equal, and corresponding side lengths grow by the same scale factor.

Problem 4.4 touches on the Angle-Angle-Angle Similarity Postulate, which was explored in *Shapes and Designs* and will be investigated further in the Grade 8 Unit *Butterflies, Pinwheels, and Wallpaper*. The Side-Side-Side Similarity Theorem is not explored in this Unit but is included here for reference.

Since Angle-Angle-Angle Similarity is considered to be a mathematical postulate, it does not need to be proven. A graphic explanation of the AAA Similarity Postulate is shown below. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete animation.

Angle-Angle-Angle Similarity Postulate

All pairs of corresponding angles are congruent, and all pairs of corresponding sides are proportional.



$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Triangle *ABC* is similar to Triangle *DEF*.

We can also use rep-tiles to illustrate the AAA Similarity Postulate. Suppose there are two triangles, A and B, which have congruent corresponding angles. We can find a third triangle, Triangle C, that has congruent corresponding angles and can nest inside A and B. We can identify the ratio between each side length in Triangle C to each side length in Triangle A as 1 : *a*. We can identify the ratio between each side length in Triangle C to each side length in Triangle B as 1 : *b*. This would mean that the ratio between side lengths in Triangle A and corresponding side lengths in Triangle B would be *a* : *b*, showing, again, that each pair of corresponding side lengths of Triangles A and B have a common scale factor.

Note: Since we know that the sum of the angles of a triangle is 180°, we need only check two pairs of corresponding angles in order to verify the similarity of two triangles. This fact is known as the Angle-Angle Similarity Postulate.

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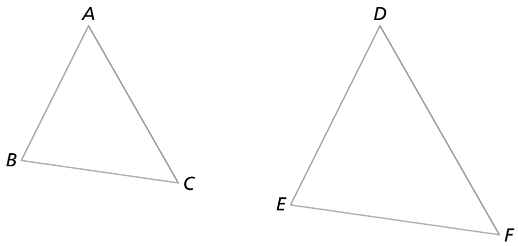
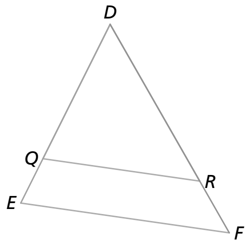
The Side-Side-Side Similarity Theorem states that, if corresponding sides of two triangles are proportional, then the triangles are similar.

Take triangles *ABC* and *DEF* to have corresponding sides that are proportional, such that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.



You can draw a line *QR* such that the line is equal in length to line *BC* and parallel to line *EF*.



Since *QR* and *EF* are parallel, angles *Q* and *E* are congruent, as are angles *R* and *F*. So, triangles *DQR* and *DEF* are similar.

Since triangles *DQR* and *DEF* are similar, then $\frac{DQ}{DE} = \frac{DR}{DF} = \frac{QR}{EF}$.

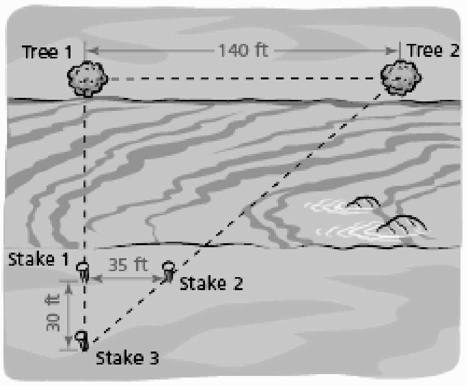
We know that *AB* must be congruent to *DQ*, and *AC* must be congruent to *DR*. These facts result from the two proportions $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ and $\frac{DQ}{DE} = \frac{DR}{DF} = \frac{QR}{EF}$, and the fact that $QR = BC$.

Since all corresponding line segments are congruent for triangles *ABC* and *DQR*, we can assume that the triangles are congruent by the Side-Side-Side Congruence Postulate. And, as similarity is transitive, since *DQR* is similar to *DEF*, its congruent triangle *ABC* must also be similar to *DEF*.

Solving Problems Using Similar Figures

Equivalent ratios can be used to solve interesting problems.

An application of similar triangles is finding an inaccessible distance, such as the distance across a river.



What is the distance across the river from Stake 1 to Tree 1?

Another group of students sketches a different diagram with similar triangles. They put their stakes in different places. Does this second group get the same measurement for the width of the river?

3.4 Out of Reach: Finding Lengths With Similar Triangles

Once students know that the triangles are similar, they can find the missing distance by using equivalent ratios of corresponding sides between the two similar triangles.

Another example of how similarity can be used to solve problems involves the use of shadows. Shadows can be thought of as corresponding sides of similar triangles, because the sunlight hits the objects at the same angle.

A tree of unknown height and a stick, both of which cast shadows, are shown below. To find the height of the tree, you can use the scale factor between the lengths of the shadows. Since the scale factor between the 6-foot stick's shadow (4.5 feet) and the tree's shadow (25 feet) is $5\frac{5}{9}$, you can multiply the height of the stick by $5\frac{5}{9}$ to obtain the height of the tree: $5\frac{5}{9} \times 6$ ft $= 33\frac{1}{3}$ ft. You can also think of this in terms of a proportion, or a pair of equivalent ratios: $\frac{x}{25} = \frac{6}{4.5}$. Finding the value of x that makes the ratios equivalent gives you the height of the tree.

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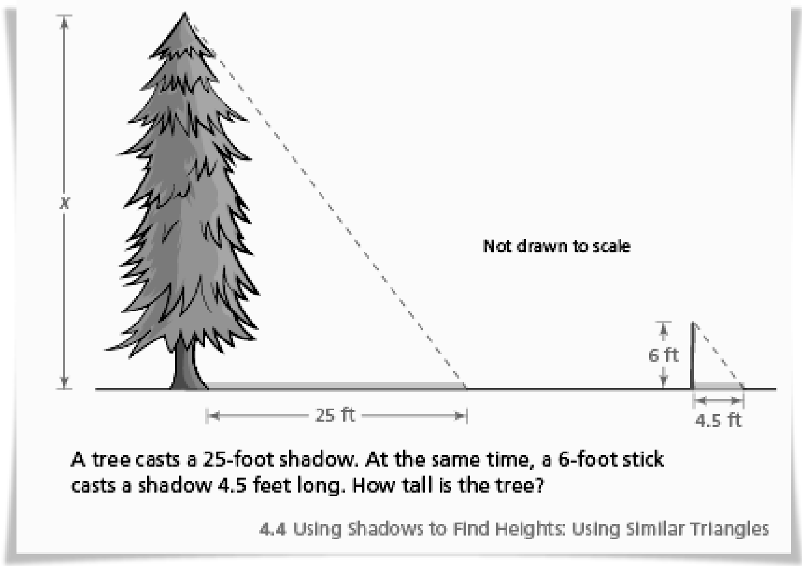

In this example, since the corresponding angles of each triangle are congruent, the two triangles are similar. Students take this as fact in this Unit, but they will explore the postulate more thoroughly in Grade 8. To see a graphic explanation of this fact, see **Similarity of Triangles**.