Mathematics Background

Scaling Ratios as a Strategy

The subtitle of *Comparing and Scaling* is *Ratios, Rates, Percents, and Proportions*. This subtitle makes clear that the heart of the Unit is to recognize when it is appropriate to make multiplicative comparisons. Throughout the Unit, students develop strategies for working with ratios, rates, percents, and proportions. They need to be able to use these strategies with understanding and efficiency.

To compare two or more related measures or counts, such as 3 roses for $5 and 7 roses for $9, you need strategies that allow the related pairs of numbers to be compared. Simple subtraction will not tell you what you want to know. You need to have an understanding of ratio and proportion. A proportion is a statement of equality of two ratios. In the example of the roses, you need to find a way to scale the ratios of 3 to 5 and 7 to 9 so that they can be directly compared.

Some students may think these two ratios are the same, reasoning that 4 has been added to each of the numbers 3 and 5 to get 7 and 9. This is a misconception about when additive comparisons are appropriate. If you appropriately scale both ratios so that either the number of roses or the costs are the same, you can make a simple multiplicative comparison of the quantities that are not the same.

Example: Scaling Denominators

If you want to scale the prices to compare the ratios, use the same sort of thinking used for finding a common denominator. Look for a number that represents a multiple of both 5 and 9.

$$\frac{3}{5} = \frac{3 \times 9}{5 \times 9} = \frac{27}{45} \quad \text{and} \quad \frac{7}{9} = \frac{7 \times 5}{9 \times 5} = \frac{35}{45}$$

$45 buys 27 roses.

$45 buys 35 roses.

You can now compare the ratios 27 roses for $45 and 35 roses for $45. The second option gives you more roses for the same amount of money.

Example: Scaling Numerators

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35} \quad \text{and} \quad \frac{7}{9} = \frac{7 \times 3}{9 \times 3} = \frac{21}{27}$$

$35 buys 21 roses.

$27 buys 21 roses.

Recall that when working with fractions, you would find common denominators to create equivalent fractions before comparing and combining them. Similar mathematical thinking is required to create equivalent ratios. As seen in the rose example, finding common numerators can work, too, depending on which unit you wish to compare.

Ratios can be written several ways.

2 to 3 — Use the word "to."

2 : 3 — Use a colon.

$\frac{2}{3}$ — Write as a fraction.

In the rose example, the convenience of writing the ratios as fractions supports the thinking needed for scaling up the ratios. Make sure, however, that students can differentiate between a ratio written as a fraction and a fraction representing the fractional part of a whole. We address this in the next section.

Using Ratio Statements to Find Fraction Statements of Comparison

The statement "the ratio of girls to boys in a class is 15 to 9" can be written as the fraction $\frac{15}{9}$, but it does not mean that the fraction of students in the class that are girls is $\frac{15}{9}$. This is confusing for students and leads some teachers to avoid using the fraction form of a ratio. You may choose to confront the confusion by asking the fraction question directly.

Maria says the fraction of the class that is girls is $\frac{15}{9}$. Bob says the fraction of the class that is girls is $\frac{15}{24}$. Who is correct and why?

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To answer the question correctly, you need to recognize that another quantity is needed.

$$15 + 9 = 24$$

15: Number of girls.
9: Number of boys.
24: Number of students.

The part-to-whole comparison is as follows.

$$15 : 24$$

15: Number of girls.
24: Number of students.

Bob is correct.

Per Quantities: Finding and Using Rates and Unit Rates

If you compute the price per rose, you will have a rate comparison for the roses problem.

Example: Unit Prices

3 roses for $5	7 roses for $9
1 rose for $1.67	1 rose for $1.29

1 rose for $1.67: Unit rate (price) for 3 roses is $5.
1 rose for $1.29: Unit rate (price) for 7 roses is $9.

Clearly, $1.29 per rose is a better price. At $1.67 per rose, 7 roses would cost $11.69. This is a different comparison with the same result.

Let's explore this strategy a bit further.

Here are two ratios that suggest rates:

Sascha goes 5 miles in 20 minutes on the first part of his bike ride. On the second part, he goes 8 miles in 24 minutes. On which part is he riding faster?

Many students will intuitively want to divide the miles and the minutes to get a result, but they may lose track of which number is divided into the other. Consequently, they produce a quotient, but have no idea what that number means in the problem. Here, the comparison can be made in two different ways by computing two different unit rates.

Suppose a student decided to divide 5 by 20 and 8 by 24. She gets the two numbers 0.25 and 0.333. What do these two numbers mean?

OR

She might have divided 20 by 5 and 24 by 8. This division gives us 4 and 3. What do these numbers mean? She needs to know before she can decide what the numbers say about the two legs of the bike ride.

Start again and this time, carry the label with the quantities.

$$\frac{5 \text{ miles}}{24 \text{ minutes}} = 0.25 \text{ miles per minute}$$

$$\frac{8 \text{ miles}}{24 \text{ minutes}} = 0.333 \text{ miles per minute}$$

Miles per minute is the unit rate for miles compared to 1 minute.

Note that the comparison is clear. The times are the same, 1 minute, and the distances can be compared. 8 miles in 24 minutes is faster.

Note that she could divide the other way, as well:

$$\frac{20 \text{ minutes}}{5 \text{ miles}} = 4 \text{ minutes per mile}$$

$$\frac{24 \text{ minutes}}{8 \text{ miles}} = 3 \text{ minutes per mile}$$

Minutes per mile is the unit rate for minutes compared to 1 mile.

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Now the lesser number means that it takes less time to travel one mile. So Sascha is riding faster during the leg of the trip where he travels 8 miles in 24 minutes.

Rate tables are very helpful for problems such as these. The units are visible, and scaling "to 1" is done in a way that makes the meaning of each division clear.

D miles	5	■	1
t minutes	20	1	■

To scale 20 minutes to get 1 minute, divide by 20. Scaling 5 miles the same way yields 0.25 miles. To scale 5 miles to get 1 mile, divide by 5. Scaling 20 minutes the same way yields 4 minutes.

D miles	5	0.25	1
t minutes	20	1	4

What makes unit rates so interesting, and somewhat difficult for students, is that there are two options when working with two numbers. The units help students think through the situations with a goal of building the flexibility to use either set of unit rates to compare the quantities.

One of the recurring themes of this Unit is representing data in different ways. Each way may tell us something that is not as obvious in other representations. The comparison in the rose example can be made in several ways: using unit rates, comparing the ratios in fraction form to determine which is greater, or scaling both rates until the price is the same or the number of roses is the same. Developing strategies for deciding what the comparison situation calls for and for making comparisons are major goals of this Unit.

Relating Ratios, Fractions, and Percents

It is often useful to change one form of a comparison statement to another. The question is, can you write a percent statement given either a ratio or a fraction statement, and can you write a ratio or fraction statement given a percent comparison statement? Let's explore this with an example.

The ratio of concentrate to water in a mix for lemonade is 3 cups concentrate to 16 cups water. The questions you might ask are "What fraction of the lemonade will be concentrate?" or "What percent of the lemonade will be concentrate?" First, find the total amount the recipe makes: 19 cups. Then write the fraction of the lemonade that is concentrate: $\frac{3}{19}$. Divide the concentrate by the total: $3 \div 19 = 0.15789\ldots$, or about 15.8% concentrate.

Example: Scaling Ratios

Suppose you know that the percent of boys in a class is 48%, and you want to write this as a ratio.

$48\% = \frac{48}{100}$ — Percent means "out of 100" so rewrite the percent as a fraction with a denominator of 100.

$\frac{48}{100} = \frac{12}{25}$ — Scale down by $\frac{4}{4}$ to find the equivalent part-to-whole ratio.

For the girls in the class, we have

$52\% = \frac{52}{100}$ — Rewrite as a fraction with a denominator of 100.

$\frac{52}{100} = \frac{13}{25}$ — Scale down by $\frac{4}{4}$ to find the equivalent part-to-whole ratio.

Note that this does not necessarily mean that there are 12 boys in the class. Instead, it means that, for every 25 students in the class, 12 are boys.

You could also obtain the ratio of girls to students without going through the above process. Use the ratio of boys to students in the class from the example above.

Example: Write an Equation

For every 25 students in a class, 12 of the students are boys. Here, a part and a whole are represented (boys and students). To express the ratio as a part-to-part ratio, you need to find the magnitude of the other part of the ratio (girls). In order find the part of class that are girls, you can use an equation with the information you know (part that are boys and whole class).

$\frac{12}{25}$ — Ratio of boys to total students.

12	+	13	=	25
boys	plus	girls	is	students

$\frac{13}{25}$ — Ratio of girls to total students.

The above ratios in turn yield the part-to-part ratio of boys to girls, which is $\frac{12}{13}$. Again, this does not necessarily meant that there are 25 students, 12 boys, or 13 girls in the class. Instead, it is a proportional comparison.

Proportions and Proportional Reasoning

The related concepts and skills in this Unit are often referred to as *proportional reasoning*. Forming ratios to make comparisons is the heart of proportional reasoning.

A *proportion* is a statement of equality between two ratios. What makes this idea powerful is that if we know one ratio is equivalent to another, but we do not know both terms of one of the ratios, we can use what we already know about scaling or finding equivalent fractions to find the unknown part. Again, let's look at an example.

> *It takes Glenda 350 steps on the elliptical machine to go 0.1 mile. When her workout is done, she has gone 3 miles. How many steps has she taken on the machine?*

Example: A proportion and a solution

Scale up by multiplying by $\frac{30}{30}$.

$$\frac{350 \text{ steps}}{0.1 \text{ mile}} = \frac{x \text{ steps}}{3 \text{ miles}}$$

$$\frac{350 \text{ steps} \times 30}{0.1 \text{ mile} \times 30} = \frac{x \text{ steps}}{3 \text{ miles}}$$

$$\frac{10{,}500 \text{ steps}}{3 \text{ miles}} = \frac{x \text{ steps}}{3 \text{ miles}}$$

Same denominators, so numerators are equal.

Glenda took 10,500 steps.

The strategy we use to find the number by which we multiply, or "scale," is the same as the thinking process we use to find common denominators for fractions.

Scaling Strategies

Finding a Scale Factor Building on strategies from *Stretching and Shrinking*, students will likely form an efficient strategy for scaling one of the ratios in a proportion to make an equivalent ratio, making it clear what the unknown part of the other ratio is. Below is Question D, part 2 from Problem 1.4. The challenge of this proportion lies in the fact that the scale factor is not immediately seen.

$\frac{7}{12} = \frac{x}{9}$ 12 times what number is 9? $12 \times \frac{9}{12} = 9$, so we need a scale factor of $\frac{9}{12}$ or 0.75.

$\frac{7 \times 0.75}{12 \times 0.75} = \frac{x}{9}$ Scale the numerator and the denominator of the left-hand side by 0.75.

$\frac{5.25}{9} = \frac{x}{9}$ Since the denominators are equal, the numerators are equal.

$5.25 = x$

The beauty of this efficient scaling strategy is that it never obscures that students are making equivalent ratios. A variation on the strategy is to scale both ratios in a proportion to have a common denominator.

Using a Unit Rate Once students have encountered rate tables, they have another tool for finding a solution of a proportion—scaling to a unit rate first.

Miles	7	$\frac{7}{12}$	x
Minutes	12	1	9

Generalizing Scale Factors Helping students to make their own reasoning explicit can lead to a generalized method of solving proportions. For example, when many students solve the proportion $\frac{3}{7} = \frac{x}{343}$, they do the following arithmetic:

$$\frac{3 \times \frac{343}{7}}{7 \times \frac{343}{7}} = \frac{x}{343}$$

The scale factor that scales 7 up to 343 is $\frac{343}{7}$.

The denominators are now equal, so the numerators are equal.

So, the students know that $3 \times \frac{343}{7} = x$.

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Consequently, for solving a general proportion $\frac{a}{b} = \frac{x}{c}$, you can follow the same reasoning. Suppose you wish to find an unknown that is a numerator. You would first calculate the scale factor from the given denominators.

$$\frac{a}{b} = \frac{x}{c} \quad \text{Scale factor is } c \div b \text{ or } \frac{c}{b}.$$

$$\frac{a \times \frac{c}{b}}{b \times \frac{c}{b}} = \frac{x}{c}$$

Scale the numerator and the denominator of the left-hand side by $\frac{c}{b}$.

$$\frac{a \times \frac{c}{b}}{c} = \frac{x}{c}$$

$$a \times \frac{c}{b} = x$$

$$\frac{a \times c}{b} = x$$

Since the denominators are now equal, the numerators must be equal too.

With the unknown in the denominator, find the scale factor using the numerators so that you can scale the denominators to find the unknown.

$$\frac{a}{b} = \frac{c}{x} \quad \text{Scale factor is } c \div a \text{ or } \frac{c}{a}.$$

$$\frac{a \times \frac{c}{a}}{b \times \frac{c}{a}} = \frac{c}{x}$$

Scale the numerator and the denominator of the left-hand side by $\frac{c}{a}$.

$$\frac{c}{b \times \frac{c}{a}} = \frac{c}{x}$$

$$b \times \frac{c}{a} = x$$

$$\frac{b \times c}{a} = x$$

Since the numerators are now equal, the denominators must also be equal.

Using Fact Families An alternative strategy can be built using fact families rather than using scaling strategies.

$$\frac{a}{b} = \frac{c}{x}$$

$$\frac{a}{b} = c \div x \quad \text{Rewrite } \frac{c}{x} \text{ as } c \div x.$$

$$x = c \div \frac{a}{b}$$

These two equations belong to the same fact family.

$$x = \frac{cb}{b} \div \frac{a}{b}$$

Rewrite using b as the least common denominator.

$$x = cb \div a$$

$$x = \frac{cb}{a}$$

Note that these are the equations you would get by cross multiplication, but here the explanation is built on students' ways of reasoning.

Using Cross Multiplication If cross multiplication is mentioned, and if the students seem interested, don't give a procedure. In a sense, cross multiplication asks the question,

> *What would be the numerator if these two fractions had a specific common denominator (the product of the original denominators)?*

Develop the idea based on what your students already know—finding common denominators.

$\frac{7}{12} = \frac{x}{9}$

If you choose a common denominator that is the product of both denominators in the proportion, you get the equation below.

$\frac{7 \times 9}{12 \times 9} = \frac{x \times 12}{9 \times 12}$

Because the denominators are equal, the numerators are equal.

$7 \times 9 = 12x$

Note that this is the same equation as the equation that would arise from cross multiplication. Basing this on finding common denominators, however, retains the concept of comparing ratios, which is the focus of *Comparing and Scaling*.

Then, solve for x using a fact family or inverse operations.

$x = \frac{7 \times 9}{12}$

Note: Students who use the scaling method for this example could correctly choose 36, 72, 144, etc. as a common denominator. Alternatively, they could use a scale factor of $\frac{9}{12}$ to write $\frac{7 \times \frac{9}{12}}{12 \times \frac{9}{12}} = \frac{x}{9}$. So, $x = \frac{7 \times 9}{12}$. Notice that this last method retains the sense of ratios being compared and is as short and efficient as cross multiplication. Cross multiplication hides the meaning of proportionality. Students who use this method may become confused when multiplying or dividing fractions, including when working with algebraic fractions and equations.

Formalizing Proportions

How far you go in formalizing the solving of proportions will depend on you and your students. We highly recommend that you do not impose solution strategies that have no meaning for the students. While cross-multiplication is efficient, for most students at this level it is used without any understanding of why it works and, consequently, is often misused. We believe that students are better served by having the time to think through situations requiring solving proportions and to develop flexibility in approaching a problem so that possible solution strategies are not missed in a rush to an algorithm.

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This approach also builds on the mathematics students already know and ways of thinking that they have already acquired. Helping students make sense of mathematics is encouraging thinking and flexibility that will allow them to feel confident in tackling problems that do not look exactly like ones they have already solved. Part of the goal of this Unit is for students to make judgments about a situation and to choose methods for comparing and for scaling.

Graphs and the Constant of Proportionality

When two variables, x and y, are proportionally related, the relationship can be represented as the proportion $\frac{y}{x} = \frac{a}{b}$, where a and b are related values of y and x. Below, we derive an equation from the above proportion.

$\frac{y}{x} = \frac{a}{b}$ $by = ax$	Scaling to the common denominator xb, we get these equal cross products.
$y = \left(\frac{a}{b}\right)x$	The expression $\frac{a}{b}$ is the unit rate.
$y = mx$	Let $m = \frac{a}{b}$. Substitute m for $\frac{a}{b}$.

The equation $y = \left(\frac{a}{b}\right)x$ or $y = mx$ is a linear relationship, which students will learn more about in *Moving Straight Ahead*.

Example: If 10 oranges cost $1.90, how much will 13 oranges cost?

x = number of oranges y = cost of x oranges	Define variables x and y.
$y = \left(\frac{a}{b}\right)x$	The expression $\frac{a}{b}$ is the unit rate.
$y = \left(\frac{1.90}{10}\right) \cdot 13$	Substitute 1.90 for a, 10 for b, and 13 for x.
$y = 0.19 \cdot 13$	Here, $m = 0.19$ is the unit rate.
$y = 2.47$	Simplify.

The cost for 13 oranges will be $2.17.

We have seen that we can use a proportion or unit rate to predict values of x or y, if we know values of a related pair of variables, a and b. We can also use the equation $y = mx$ to predict values of x or y.

For example, in Problem 2.2, students use a rate table to find the cost of one pizza at Royal Pizza, where 10 pizzas cost $120. They can predict the cost of any number of pizzas, say 9 pizzas, by extending the rate table

Number of Pizzas	1	2	3	4	5	6	7	8	9	10
Price	$12	$24	$36	$48	$60	$72	$84	$96	$108	$120

or by setting up a proportion.

$$\frac{120}{10} = \frac{C}{9}$$

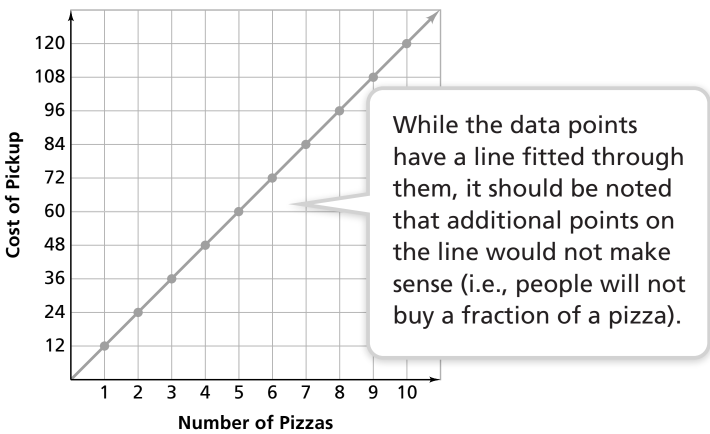
Cost is in the numerator.

Number of pizzas is in the denominator.

They can also use the equation $C = 12n$. The coefficient of n in this equation is the unit rate per pizza; this is also called the *constant of proportionality*.

$C = 12n$	12 is the constant of proportionality.
$C = 12 \cdot 9$	Substitute 9 for n.
$C = 108$	Simplify.

The graph of the equation represents all the pairs of values that fit the equation $C = 12n$. Points on the graph represent all the solutions for the proportion $\frac{\$120}{10 \text{ pizzas}} = \frac{\$C}{n \text{ pizzas}}$; if you know the value of either n or C, you can find the corresponding solution on this graph.



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Notice that the graph passes through (0, 0), which makes sense for cost and number of pizzas. In fact, for any relationship represented by an equation of the form $y = mx$, the graph will be a straight line passing through (0, 0); this is one of the ways students can recognize a proportional relationship from its graph.

Comparing Multiplicative and Additive Relationships

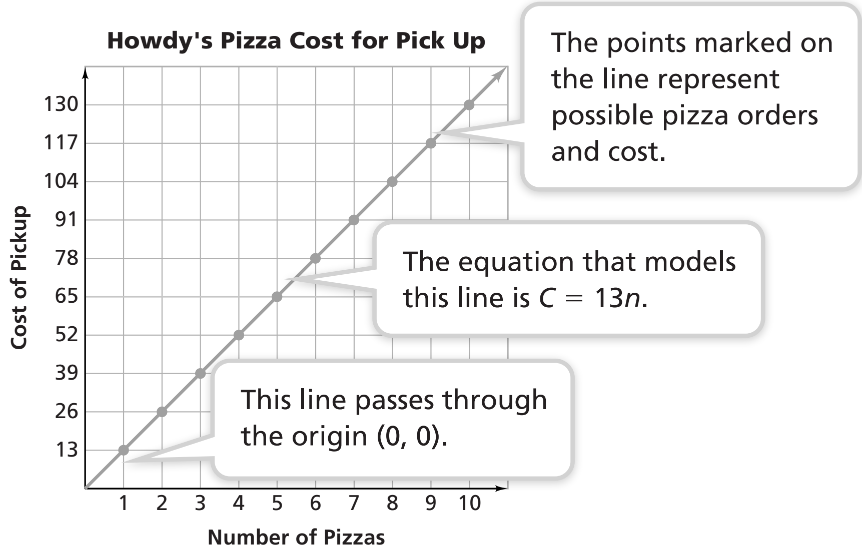
The table representing a proportional relationship also has distinguishing characteristics. In Problem 2.2, students compare tables for the costs of pizzas with and without a fixed delivery cost.

Number of Pizzas	Howdy's (pick up)	Howdy's (delivery)
1	$13	$18
2	$26	$31
3	$39	$44
4	$52	$57
5	$65	$70
6	$78	$83
7	$91	$96
8	$104	$109
9	$117	$122
10	$130	$135

In this column, while we can see a difference of 13 for each cost, these values are all 5 more than multiples of 13.

For Howdy's Pizza, we can see that the scale factor is 13, given the multiples of 13 in this column.

Below is a graph of the cost for pick-up at Howdy's Pizza.



Below is a graph of the cost for delivery at for Howdy's.