Mathematics Background

The Meaning of Probability

The terms *chance* and *probability* apply to situations that have uncertain outcomes on individual trials but a regular pattern of outcomes over many trials. For example, when you toss a coin, you are uncertain whether it will come up heads or tails. But you do know that over the long run, if it is a fair coin, you will get about half heads and half tails. This does not mean you won't get several heads in a row, or that if you get heads now you are more likely to get tails on the next toss. Uncertainty on an individual outcome but predictable regularity in the long run is a difficult concept for students to grasp. It often takes a significant amount of time and a variety of experiences that challenge prior conceptions before students understand this basic concept of probability.

The *Common Core State Standards for Mathematics* (CCSSM) calls for probability during the middle grades only in Grade 7. Consequently, this is the only Unit in the *Connected Mathematics* curriculum that develops students' abilities to understand and reason about probability. This Unit combines Investigations from the CMP2 probability Units Grade 6 *How Likely Is It?* and Grade 7 *What Do You Expect?*

The Investigations selected for this CMP3 Unit cover the CCSSM requirements. Additionally, this Unit is both accessible and interesting to Grade 7 students. From study of the Unit, students will gain an understanding of experimental and theoretical probabilities and the relationship between them. Through the examples and Problems in the Unit, students also make important connections between probability and rational numbers, geometry, statistics, science, and business.

Experimental vs. Theoretical Probability

Experimental probability is probability that is determined through experimentation. The experimental probability is the ratio of the number of times a favorable outcome occurs to the total number of trials.

> **Example**
> You can find the experimental probability of getting a head when you toss a coin by tossing a coin 20 times and keeping track of the outcomes. So, if you get 12 heads out of 20 flips of the coin, your experimental probability is $\frac{12}{20}$, or $\frac{3}{5}$, or 60%.

Theoretical probability is probability obtained by analyzing a situation. If all of the outcomes are equally likely, you can find a theoretical probability of an event by listing all of the possible outcomes. Then you find the ratio of the number of outcomes producing the desired event to the total number of outcomes.

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> **Example**
> When rolling two number cubes, there are a total of 36 possible outcomes. Of these, six outcomes have a sum of 7, so the probability of rolling a 7 on two number cubes is $\frac{6}{36}$, or $\frac{1}{6}$, or about 16.7%.

In some situations, it is easier to find theoretical probabilities. In others, it is easier to find experimental probabilities. Probabilities are useful for predicting what will happen over the long run, yet a theoretical or experimental probability does not tell us exactly what will happen. For example, if you toss a coin 40 times, you may not get exactly 20 heads. However, if you toss a coin 1,000 times, the fraction of heads will be fairly close to $\frac{1}{2}$.

In this Unit, students will find experimental probabilities for a tossed paper cup landing on its side or on one of its ends. They will not be able to determine the theoretical probabilities. Although "end" and "side" are the possible outcomes, they are not necessarily equally likely. If you toss the cup many times, you can use the ratio of the number of times the cup lands on its side to the total number of tosses to estimate the likelihood that the cup will land on its side. Since you find this ratio through experimentation, the ratio is called an experimental probability.

Statisticians have additional names for these ideas. The number of favorable outcomes, in this case the number of times the cup lands on its side, is also called the relative frequency of favorable outcomes. Statisticians also refer to particular outcomes as *outcomes of interest*. The experimental probability, the ratio of favorable outcomes to total outcomes, is also called relative frequency of favorable outcomes (relative frequency of outcomes of interest).

The experimental probability (relative frequency) that the cup will land on its side can be expressed as

$$P(\text{side}) = \frac{\text{number of times the cup lands on its side}}{\text{total number of tosses}}$$

Many uses of probability in daily life are based on experimental probabilities. You collect data for a large number of trials and observe the frequency of a particular result. This is the relative-frequency interpretation of probability. The probability that it will rain and the probability that a basketball player will make a free throw are two uses of experimental probabilities based on relative frequencies.

Another way to determine probability is to find the theoretical probability of a situation. For example, you can examine the theoretical probability of a fair coin landing heads or tails by analyzing the situation. If you toss a fair coin, you know that it will land either heads up or tails up and that each outcome is equally likely. Since there are two possible equally likely outcomes, the probability of a fair coin landing heads up is 1 of 2, or $\frac{1}{2}$. You can write this statement as $P(\text{heads}) = \frac{1}{2}$. In general, the theoretical probability that a coin will land heads up can be expressed as:

$$P(\textit{heads}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$
$$= \frac{1(\text{heads})}{2(\text{number of outcomes})}$$

Another example of a theoretical probability that occurs in this Unit involves the roll of a number cube. When you roll a number cube, there are six possible outcomes: 1, 2, 3, 4, 5, and 6. Each outcome is equally likely on any roll of a number cube. Thus, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$. If you roll a number cube 36 times, you can expect each number to occur about 6 times. You can use this theoretical probability to make an estimate: If a number cube is rolled many times, you can expect each number to occur about $\frac{1}{6}$ of the time. You can also compute the probability of events that include more than one equally likely outcome.

Take the following question, for example:

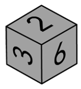
- What is the theoretical probability of rolling a multiple of 3 on a number cube?

Since two of the six equally likely outcomes, 3 and 6, are multiples of 3, the probability of a multiple of 3 occurring is $\frac{2}{6}$, or $\frac{1}{3}$.

Some important aspects of the concept of probability are illustrated below using the action of rolling a number cube. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.

Some Key Ideas About Probability

- A probability is a number that is less than or equal to 1 and greater than or equal to 0.
- In the case of rolling a number cube, there are 6 outcomes, each with a probability of $\frac{1}{6}$.
  The sum of the probabilities of all outcomes is equal to 1.
- The sum is $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$.
- Some problems involve the probability that both outcomes A **and** B occur.
  For example, $P(>3 \textbf{ and } \text{even}) = \frac{2}{6}$ or $\frac{1}{3}$.
- Some problems involve the probability that outcome A **or** outcome B (but not both) occurs.
  These are called mutually exclusive events.
  For example, $P(1 \textbf{ or } \text{prime}) = P(1) + P(\text{prime}) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6}$, or $\frac{2}{3}$.
- In general, for mutually exclusive events, $P(\text{A} \textbf{ or } \text{B}) = P(\text{A}) + P(\text{B})$.
- Some problems involve the probability that either outcome A or outcome B (or both) occurs.
  These events are *not* mutually exclusive.
  For example, $P(>2 \textbf{ or } \text{even}) = P(>2) + P(\text{even}) - P(>2 \textbf{ and } \text{even}) = \frac{4}{6} + \frac{3}{6} - \frac{2}{6} = \frac{5}{6}$.
- In general, for events that are not mutually exclusive, $P(\text{A} \textbf{ or } \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$.
- The probability of rolling a 7 is $P(7) = \frac{0}{6}$, or 0.
- The probability of rolling a number less than 10 is $P(<10) = \frac{6}{6}$, or 1.
- The probability of rolling a number that is *not* 4 is $P(\textbf{not } 4) = 1 - P(4) = 1 - \frac{1}{6} = \frac{5}{6}$.
- In general, $P(\textbf{not } \text{A}) = 1 - P(\text{A})$.



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> In mathematics, *random* has a meaning somewhat different from its everyday usage. In everyday English, *random* is often used to mean haphazard and completely unpredictable, in either the long or short term.
>
> In mathematics, *random* means that any particular outcome is unpredictable, but the long-term behavior is quite stable. When you toss a coin, it is random because you never know whether the next toss will be heads or tails, but you do know that in the long run you will have close to 50% heads.

Mathematically, an *outcome* is one of the possible results of an experiment or event. Mathematicians also use the term *result* to mean *outcome*.

For example, consider the probability of heads occurring when a coin is tossed. The actual tossing of the coin is the event. In this event, heads occurring is the result or outcome.

In this Unit, we have chosen the language of *outcome* because it is more intuitive for both teachers and students. On occasion, *result* is used if it is more natural to do so. For most purposes, *outcome* and *result* are interchangeable terms.

General Strategies for Finding Outcomes

In many situations, making an organized list can help you determine all the possible outcomes. In the situation of rolling a number cube, there are only six outcomes to list. Some situations involve more than one action.

For example, suppose you toss a fair coin twice. How many possible outcomes are there? You can list the outcomes as they come to mind, but it is often more efficient to generate the outcomes in a systematic way. This helps to ensure that you find all the possible outcomes.

In the situation of tossing a coin twice, you can list the possibilities for the first toss, namely heads (H) or tails (T). Suppose the first toss resulted in heads (H). What can happen next?

Since the two tosses of the coin are independent (the results of one do not affect or depend on the other), you have two possible outcomes (H or T) for the second toss.

When the first toss is H, there are two possible outcomes for the second toss: H or T. Here, you have HH or HT.

First Toss	Second Toss
H	H
H	T
T	H
T	T

When the first toss is T, there are two possible outcomes for the second toss: H or T. Here you have TH and TT.

Thus, there are four possible outcomes when you toss a coin twice. Since you have considered all the possibilities in a systematic way, you can feel confident that you have found all of the possible outcomes.

Note: When students toss two coins at once, they may perceive only one way to get a no-match: one heads and one tails. Because the two coins are the same, students may not see heads-tails as being different from tails-heads. One way to address this is to toss one coin twice, paying attention to the order that the heads and tails come up. Not all students will see this as being the same as tossing two coins at the same time. Another way to investigate the question is to have the coins have different years stamped on them. A 1975 head and 1982 tail are different from a 1975 tail and 1982 head.

The string HHHHHHHHHHHHHHHHHHHH is just as likely as any other string of 20 tosses, such as HTTHHHHTTHTHTHTHTHTT. Since there are two choices for the first position, two for the second position, and so on for all 20 positions, there are $2 \times 2 \times 2 \times 2 \ldots$ (a product of 20 factors of 2), or 2^{20}, different strings that can occur. A string of 20 heads is one of these 1,048,576 possible strings.

Notice that there is only one way to have 20 heads in a row, but there are over a million ways to have a mixture of heads and tails. For example, there are 184,756 ways to get 10 heads and 10 tails. Hence, having some mixture of heads and tails is much more likely than having 20 heads, because there are more ways to arrange them. Still, any one specific arrangement of 10 heads and 10 tails is just as likely (or unlikely) as a string of 20 heads. The probability of each specific string is $(0.5)^{20}$ or $\frac{1}{1,048,576}$ (roughly one in a million).

Law of Large Numbers

Experimental data gathered over many trials should produce probabilities that are close to the theoretical probabilities. This idea is sometimes called the *Law of Large Numbers*. If you can calculate a theoretical probability, you can use it to predict what will happen in the long run rather than relying on experimentation.

The Law of Large Numbers applies to mathematically random outcomes. It is important to understand what the Law of Large Numbers says, as well as what it does not say. It does not say that you should expect exactly 50% heads in any given large number of trials. Instead, it says that as the number of trials gets larger, you expect the percent of heads to get closer to 50%. For 1 million tosses, exactly 50% (500,000) heads is improbable. But for 1 million tosses, it would be extremely unlikely for the percent of heads to be less than 49% or more than 51%.

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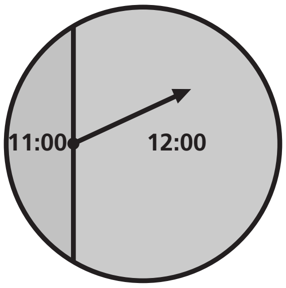
The probabilities in a spinner are determined by the relative measures of the angles in each section rather than by their areas. The two are interchangeable in the spinners of Investigation 3 because the center of the spinner is in the center of a circle. The areas of the sections vary in proportion to their angles.

However, in the spinner below the two outcomes are still equally likely, although their areas are not the same. The angles taken up by each section are each equal to 180°.



Using Probabilities to Make Predictions and Decisions

Once you have a probability (theoretical or experimental), you can use it to make predictions. For example, if a coin is tossed 1,000 times, you can predict that heads will occur about 500 times. If you roll a number cube 1,000 times, you can predict that a 3 occurs about $\frac{1}{6}$ of the time, or about $\frac{1}{6} \times 1{,}000$ or 167 times.

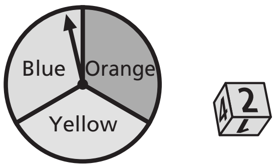
Another way to think of this is as equivalent fractions: $\frac{1}{6} \approx \frac{167}{1{,}000}$. Note that $\frac{167}{1{,}000}$ is approximately equal to $\frac{1}{6}$.

Students often seek ways to make decisions that are fair. For example, how can you select students for a field trip that can accommodate only ten students? To be fair, the method chosen should give each student an equal (or the same) chance of being chosen.

Also, students sometimes find themselves in situations in which they would like to know the probabilities of a favorable outcome, such as rubbing two spots on a card containing five spots. Some of the spots, if rubbed, lead to prizes. Such situations can be simulated by an experiment such as choosing colored marbles from a bag. Knowing the probability in these situations can help make decisions about whether or not to play the game.

Theoretical Probability Models: Tree Diagrams and Organized Lists

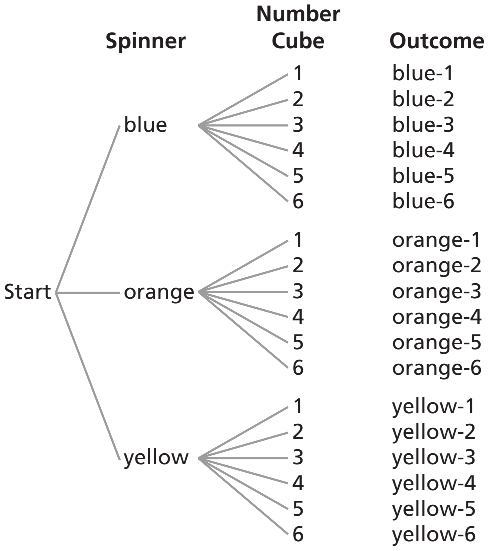
Tree diagrams, introduced in Investigation 2, offer students another way to systematically determine all the possible outcomes in a situation. For example, suppose you spin the pointer of a spinner with three sections (made by three angles with the same measure) and you roll a number cube. An organized list or a tree diagram can be used to determine all the possible outcomes.



Organized List

Color	Number Cube	Outcome
blue	1	blue-1
blue	2	blue-2
blue	3	blue-3
blue	4	blue-4
blue	5	blue-5
blue	6	blue-6
orange	1	orange-1
orange	2	orange-2
orange	3	orange-3
orange	4	orange-4
orange	5	orange-5
orange	6	orange-6
yellow	1	yellow-1
yellow	2	yellow-2
yellow	3	yellow-3
yellow	4	yellow-4
yellow	5	yellow-5
yellow	6	yellow-6

Tree Diagram



Since the number cube is fair, each of the numbers 1–6 has the same probability of occurrence, $\frac{1}{6}$. Since the spinner is divided into three angles of the same size, each color has the same probability of occurrence, $\frac{1}{3}$. Thus, each color and number combination is equally likely, and since there are 18 combinations, each has a probability of $\frac{1}{18}$.

In this Unit, students use tree diagrams to find the number of equally likely outcomes in situations with a large number of possible outcomes. Tree diagrams are particularly useful for listing outcomes in situations involving a series of actions, such as rolling a number cube twice, tossing a coin four times, or choosing several items from a menu, such as a sandwich, drink, and dessert.

Tree diagrams can be used as a basis for understanding the multiplication of probabilities, though they are not intended to be used that way in this Unit. Students do not yet understand enough about probability to know when and why it is appropriate to multiply probabilities.

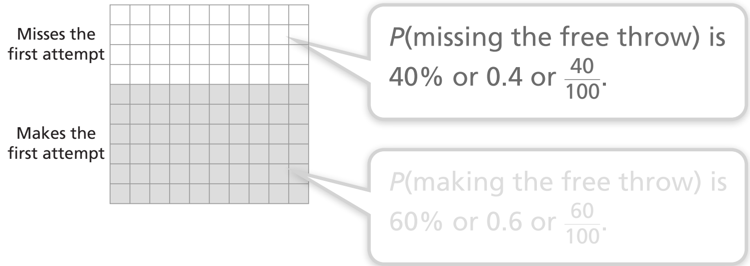
For example, in the preceding example, the probability of spinning a blue is $\frac{1}{3}$ and the probability of rolling a 1 is $\frac{1}{6}$. The probability of spinning a blue and rolling a 1 is $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$.

Theoretical Probability Models: Area Models

Area models, like tree diagrams, are useful for finding probabilities in situations involving successive events, such as when a basketball player is allowed to attempt a second free throw only if his or her first free throw succeeds. Unlike tree diagrams, area models are particularly powerful in situations in which the possible outcomes are not equally likely.

The following steps demonstrate how to create an area model to show the probability that Nishi, a basketball player who makes 60% of her free throws, will score 0, 1, or 2 points in a two-attempt free throw situation in basketball. In a two-attempt situation, the player will get to attempt a second free throw whether or not her first free throw succeeds.

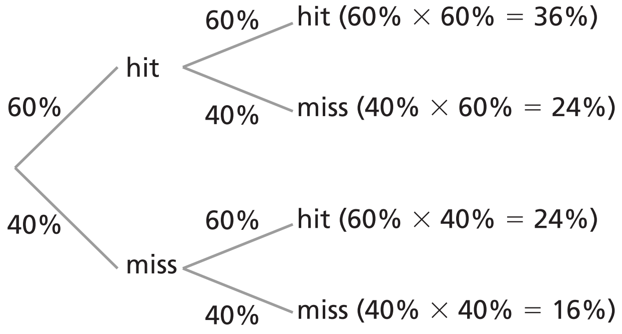
The first free throw has two possible outcomes, making or missing the free throw.



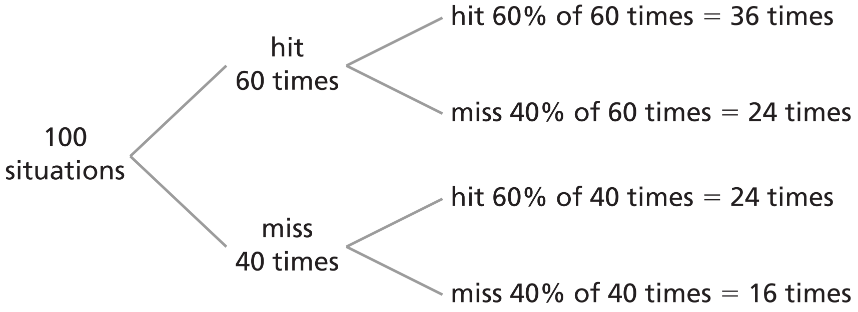
P(missing the free throw) is 40% or 0.4 or $\frac{40}{100}$.

P(making the free throw) is 60% or 0.6 or $\frac{60}{100}$.

To use a tree diagram approach in a situation in which outcomes are not equally likely, each branch of the tree must be weighted by the probability that it will be chosen. This idea is quite difficult for students at this stage to understand; they have used tree diagrams only in situations involving equally likely outcomes. An area analysis makes the weighting more obvious. It is not recommended that you introduce this idea to your students now, but shown here is a tree diagram that works.

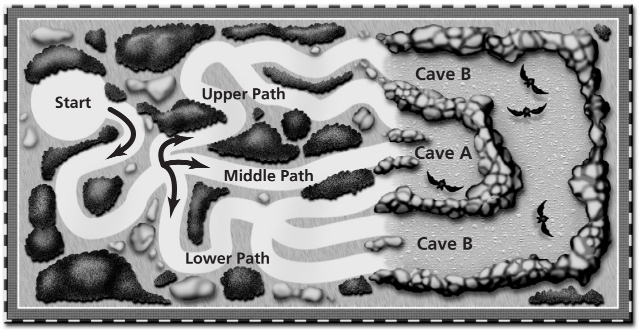


Students will sometimes make a modified version of a weighted tree diagram. Such a student might choose a large number of situations (100), then indicate how many of these he would expect to occur on each first branch (60 and 40, corresponding to Nishi's percent of making and missing free throws). Then each of these numbers is broken down proportionately for the next stage. In effect, this is the same idea as above, but is more accessible to students at this stage of their study of probability.



In Problem 4.3, students explore the probabilities of getting a score of 0, 1, or 2 for a basketball player who makes 60% of her free throws in a one-and-one situation.

Consider one more example of these ideas. In the ACE Exercises, students consider a path game (below) in which a player chooses a path at random at each intersection. Students are asked to figure out the probability of landing in either Cave A or Cave B.



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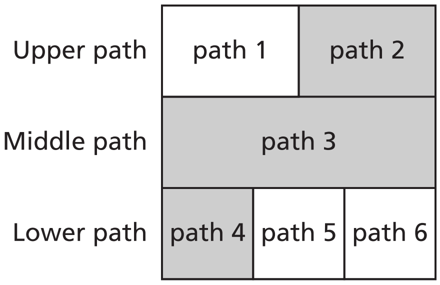
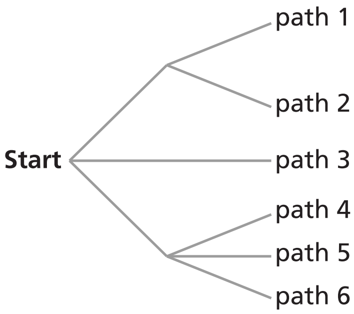
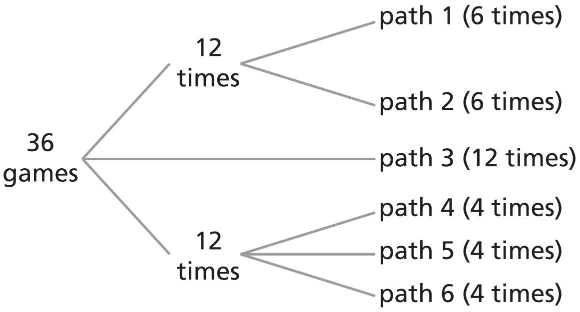
The area model for this game is first split into thirds to indicate the three equally likely paths at the first intersection: the upper path, the middle path, and the lower path. Then each of these thirds is split according to the later intersections (if any), resulting in the model below.



From the area model, it is clear that the 6 paths are not equally likely. Path 3, for instance, has a probability of $\frac{1}{3}$, while path 4 has probability of $\frac{1}{9}$. A simple tree diagram would not show this:



But the modified tree diagram described above would in fact represent the differences in the probabilities for each path. Path 3 occurs 12 out of the 36 games, more than any of the other paths.



Compound Events

Compound events are made of two or more simple events, A and B, that happen at the same time or in sequence. Sometimes these events are referred to as multistage events, which implies that it takes more than one action to create an outcome. The actions do not always have to be sequential, however. Some examples of compound or multistage events include rolling two or more number cubes, flipping two or more coins, spinning two or more spinners, or combinations of those events. In each of these cases the simple events can be done simultaneously or sequentially without affecting the possible outcomes. Sometimes the simple events can only be done in sequence, for example, by taking two basketball free throws or by choosing paths in a maze. In all the examples discussed here the simple events are independent; that is, the outcome of the first event has no influence on the probability of the outcome of the second event.

Suppose you toss two coins at the same time and are interested in finding the probability that you will get a match. There are two ways that a match can happen. You can get two heads or two tails. Let the event A represent a match. Then the probability of *A*, *P*(*A*), is the sum of the probabilities of each outcome where two coins match.

$$P(\text{A}) = P(\text{T, T}) + P(\text{H, H}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

In the example of the compound event above, we could also think of event A as a two-stage event by flipping one coin two times. It takes the toss of two coins (or one coin, twice) to get an outcome. The possibilities are (T, T), (T, H), (H, T), and (H, H).

The question is *What is the probability of each of these outcomes?* If the coin is fair, then each coin toss has a probability of landing tails or heads. The coin tosses are independent of each other. How the coin lands on a given toss is not affected by any previous toss. Here $P(\text{T, T}) = P(\text{T}) \times P(\text{T}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The same is true for each of the four possible outcomes.

For a basketball player who makes 60% of her free throws in a one-and-one situation, whether or not the player gets to take a second free throw depends on the result of the first free throw. Here the second free throw is dependent on the result of the first free throw.

Thus, *P*(0 points) can only be gotten in one way, and that is to miss the first free throw. The probability of a miss on the first free throw is 0.4. Thus, *P*(0 points) = 0.4.

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There are two possible outcomes resulting from a hit on the first free throw; the player can hit or miss the second free throw. So we have

$P(\text{2 points}) = P(\text{H, H}) = 0.6 \times 0.6 = 0.36$

$P(\text{H, M}) = P(\text{1 point}) = 0.6 \times 0.4 = 0.24$

In a one-and-one free throw situation, we have three possible outcomes, one of which is a one-stage event (0 points) and two of which are two-stage events (1 point or 2 points). The deciding factor is that the second action is dependent on the result of the first action. The sum of all possible outcomes is

$P(\text{0 points}) + P(\text{1 point})\ P(\text{2 points}) = 0.4 + 0.24 + 0.36 = 1$

As students use an area model to make sense of two-stage probability situations, take any opportunity to help those who seem ready to see the connection to multiplying probabilities. For example, in the preceding 60% two free-throw situation,

$$P(\text{a score of 0}) = \frac{40}{100} \times \frac{40}{100} = \frac{16}{100}$$

$$P(\text{a score of 1}) = \frac{60}{100} \times \frac{40}{100} + \frac{40}{100} \times \frac{60}{100} = \frac{48}{100}$$

$$P(\text{a score of 2}) = \frac{60}{100} \times \frac{60}{100} = \frac{36}{100}$$

As an area model is also used to develop an understanding of the multiplication of fractions, many students will see this connection naturally.

Expected Value

The long-term aspect of probability is a powerful concept. Rather than guarantee what will happen on a particular trial, or even in the short run, *probability models* predict what will happen in the long run over many trials. Often, this is the most valuable information we can gain about a probability situation: A prediction of the expected value of the situation. The expected value is a long-term average of the payoff of each outcome weighted by its probability.

Expected value goes beyond basic probabilities. It uses values, such as points earned in a game or money won in a contest, to weight each possible outcome by then computing the average points or dollars we can expect per game or contest in the long run. You can think of expected value as a "weighted average."

In this Unit, students are introduced to expected value in an informal yet concrete way. We do not expect them to develop a formal definition of expected value or to use a formula for finding it. In fact, students might never use the term *expected value* in their work in this Unit, instead thinking of the concept as "what is expected in the long run." However, *expected value* is vocabulary that the student text uses frequently once it is introduced.

> **Example**
>
> Consider the long-term average or expected value for a basketball player with a 60% free throw percentage in one-and-one free-throw situations.
>
> If the player goes to the line 100 times, then he or she expects a score of 0 to occur 16 times for a total of 0 points, a score of 1 to occur 24 times for a total of 24 points, and a score of 2 to occur 36 times for a total of 72 points. The total number of points expected in 100 situations is 0 + 24 + 72 = 96 points. Here the weight factor is the probability of obtaining that result. The average number of points expected in 100 trials is 96 points ÷ 100 trials, or 0.96 points per trial.
>
> We could also arrive at this result by the computation
>
> $$\frac{16}{100}(0) + \frac{24}{100}(1) + \frac{36}{100}(2) = \frac{96}{100} = 0.96$$

The question of what the free throw average is for a player with an expected value of exactly 1 point per one-and-one trip turns out to have a surprising answer. The following is a mathematical analysis that shows how problems such as this one may be revisited in high school when students are ready to solve quadratic equations.

Let p represent the probability that a player will make the free throw. Then, $(1 - p)$ represents the probability that the player will miss the free throw.

Thus, the probability of
- making 0 points is $(1 - p)$
- making 1 point is $p(1 - p)$
- making 2 points is $p \times p = p^2$.

The expected value is thus:
$P(2\text{ points}) \times 2 + P(1\text{ point}) \times 1 + P(0\text{ points}) \times 0$

Symbolically,

$$\begin{aligned} p^2 \times 2 + p(1 - p) \times 1 + (1 - p) \times 0 &= 1 \\ 2p^2 + p - p^2 + 0 &= 1 \\ p^2 + p &= 1 \end{aligned}$$

Using the Quadratic Formula to solve the resulting quadratic equation $p^2 + p - 1 = 0$, yields:

$$p = \frac{-1 + \sqrt{1 + 4}}{2} = \frac{-1 + \sqrt{5}}{2}$$

This is the Golden Ratio, which is approximately 0.6180339887. The Golden Ratio is the proportion of length to width of a rectangle that many people consider to be the most beautiful rectangle. Many ancient Greek buildings were built with facades that incorporate this ratio.

Independent and Dependent Events

Please note that the terms *independent* and *dependent* events are not mentioned in this Unit. Naming these ideas can wait until a later course in probability. In this Unit, students need only to make sense of each situation and apply the appropriate probability at each stage.

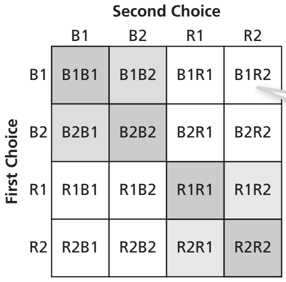
The idea of *independent* and *dependent* events is introduced informally. A more formal approach is often a major focus of probability study in high school and college courses. Yet, we feel it is important to introduce this concept because many students working through a basic probability Unit such as this one develop the belief that all events are independent.

> **Example**
>
> Suppose you twice choose a marble from a bag containing two red marbles and two blue marbles. If you replace the chosen marble after the first choice, the two choices will be independent of each other, because what you choose the first time will not affect what you choose the second time.
>
> If you do not replace the chosen marble, the second choice will be dependent on the first choice, because the probability of choosing each color the second time depends on the color chosen on the first choice. For example, if you choose a red marble the first time and do not replace it, the probability of choosing a red marble the second time is $\frac{1}{3}$ rather than $\frac{1}{2}$. Yet if you had chosen a blue marble the first time, the probability of choosing red the second time would be $\frac{2}{3}$. It is in this sense that the probability of choosing a red on the second choice is a dependent probability.

In this Unit, students analyze dependent events by using the situation to help make sense of the sequence of actions. They look at the context and determine the sequence of actions and the possibilities at each step in the sequence. The steps in the sequence guide the apportioning of the total area in an area model or the designing of a tree diagram representing all possible outcomes. Then each portion of area in an area model, or each path on a tree diagram, is compared to the total area or the total number of possible outcomes to form probability statements.

Example

Consider an area model for the marbles without replacement:



	B1	B2	R1	R2
B1	B1B1	B1B2	B1R1	B1R2
B2	B2B1	B2B2	B2R1	B2R2
R1	R1B1	R1B2	R1R1	R1R2
R2	R2B1	R2B2	R2R1	R2R2

Since you are drawing marbles *without* replacement, the outcomes B1B1, B2B2, R1R1 and R2R2 are impossible. We actually have 12 possible outcomes here.

The probability of choosing two reds is $\frac{1}{6}$. Note that the probability of choosing a red on the second choice is greater if blue was chosen on the first choice.

Binomial Events and Pascal's Triangle

Many interesting probability situations are of the type where there are exactly two possible outcomes: yes or no, boy or girl, true or false, heads or tails, and so forth. These events are equally likely. Other events such as rain or no rain, correct or wrong (on a multiple choice test), and hit or miss also have two possible outcomes. These examples are not equally likely. Events that have exactly two outcomes (not necessarily equally likely) are called *binomial events*.

If students guess at every answer for a five-item true/false quiz, there are 32 ways to answer the quiz, but only one of them has all five answers correct. The probability of getting all five answers correct is $\frac{1}{32}$.

A similar situation involves the families in the town of Ortonville from Problem 5.2. Each family has exactly five children and they all agree to give their children the same names. There are 32 ways to arrange five children according to numbers of boys and girls (BBGGG, BGBGG, GGGGG, and so forth). The probability of a family having exactly five girls is $\frac{1}{32}$. The probability of having two boys and three girls in any order is $\frac{10}{32}$.

continued on next page

Once one binomial situation has been analyzed, it is easy to analyze another binomial situation.

Pascal's Triangle is used to analyze binomial probabilities. The triangle of numbers is named after the seventeenth-century mathematician Blaise Pascal, because he was the first to name it. However, the array was in existence long before this. The first five rows are below. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Coin	True/False Test
Tossing 1 coin	1 question
Tossing 2 coins	2 questions
Tossing 3 coins	3 questions
Tossing 4 coins	4 questions
Tossing 5 coins	5 questions

Pascal's Triangle is presented only in an ACE Exercise, but students recognize the similarity between the binomial situations and can use previous results to analyze a new situation.

> **Example**
>
> In a World Series between the evenly matched Gazelles (G) and Bobcats (B), the Gazelles have won the first two games. What is the probability that the series will end in four games? Five games? Six games? Seven games?
>
> To answer these questions, students analyze the possible outcomes of the last five games. Again, there are 32 outcomes. The probability of ending in 4, 5, 6, or 7 games is equal $\left(\frac{1}{4}\text{ each}\right)$. However, the Gazelles have a greater chance of winning the series.