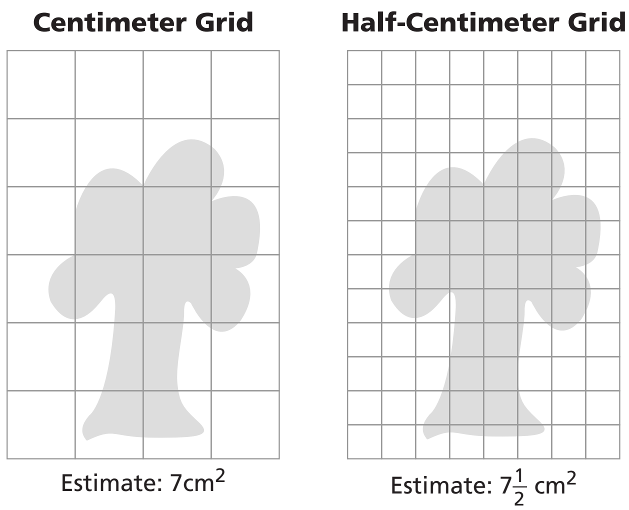
Mathematics Background

Measurement

All measurements are approximations. In their work in this Unit, students explore ways to find measures for two- and three-dimensional figures. Even using exact formulas depends on how students measure various dimensions. Any measure of length involves some estimation. If students are doing the actual measuring, they must choose their units carefully. Using smaller units of measure can result in more accurate approximations. Once the unit of length has been selected, it can be subdivided to make more accurate measurements because there is less room for estimation error.

For example, suppose you want to measure the area of an object. First, you need to select a corresponding square unit with which to cover the surface. Just as with length, using smaller units for measuring area results in less estimation error and gives more accurate results. In the following pictures, the object is measured first with centimeter squares and then with half-centimeter squares.



From the pictures, you can see more of the tree is covered with complete half-centimeter squares, leaving less room for error around the edges.

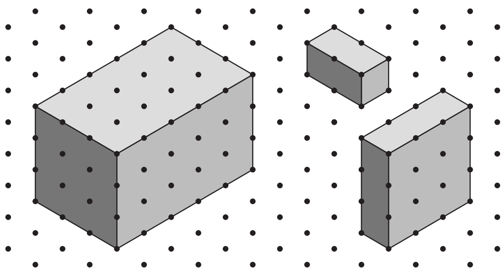
In addition, students' calculations of surface area and volume will often involve an approximation of the number π, and they will often use a calculated amount as a value in a subsequent calculation. Be aware that students' answers may differ; so multiple answers may reflect correct reasoning and correct mathematics.

Note: As in this example, many surfaces are irregularly shaped. There is no formula for calculating the area of an irregular shape. The area is determined by counting the square units that cover the surface. In this Unit, students will study figures with regular shapes, which have formulas for finding surface area and volume. All of these formulas come from strategies for covering a surface with unit squares or filling an object with unit cubes.

Drawing Three-Dimensional Figures

The conventional strategies for representing three-dimensional figures with two-dimensional drawings are not trivial learning tasks for students. It is particularly hard to give written directions for how to make such drawings, and the conventions are not generally truly faithful to perspective drawing. We suggest that you take opportunities throughout the Unit to help students learn how to make conventional drawings of three-dimensional shapes, such as in Problem 1.2 and in other convenient places later in the Unit.

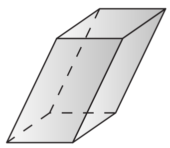
Isometric drawings are a useful two-dimensional representation of three-dimensional objects. You may wish to have isometric dot paper available throughout the Unit for students to use in making sketches of three-dimensional figures. The following figures are right rectangular prisms drawn on isometric dot paper.



Note: There is a CMP Unit, *Ruins of Montarek*, still available in print if you want to give students a solid background in making drawings of such structures built from cubes.

Rectangular Prisms

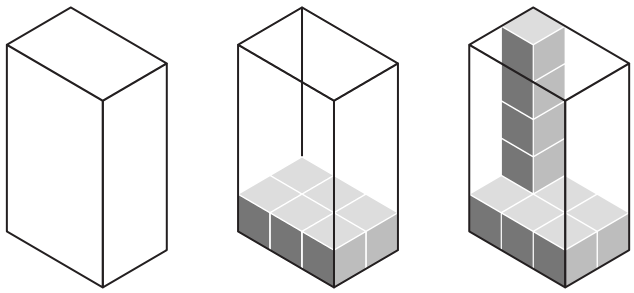
The most common rectangular prisms are three-dimensional shapes with six rectangular faces. Technically, this defines a right rectangular prism. The *Common Core State Standards for Mathematics (CCSSM)* call for students to develop an understanding of surface area and volume of right rectangular prisms in Grade 6. In this Unit, students review those ideas as a foundation for study of other right prisms. However, it is interesting to note that there are other types of prisms. An **oblique rectangular prism** also has two bases that are rectangles, but at least two of the lateral faces must be nonrectangular parallelograms. The strategies for finding the surface area and volume of an oblique rectangular prism are the same as those for a right rectangular prism.



Oblique rectangular prism

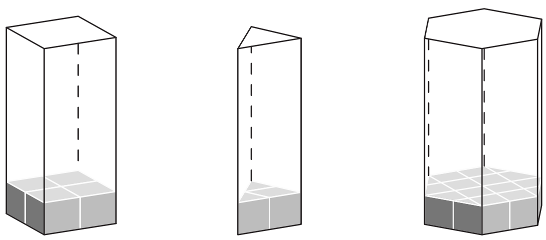
Volume of a Prism

A productive and generalizable strategy for finding the volume of a rectangular box is to count the number of layers of unit cubes it takes to fill the container. The number of unit cubes in a layer is equal to the area of the base—one unit cube sits on each square unit in the base. The volume (the total number of unit cubes) of a rectangular prism is the area of its base (the number of unit cubes in the first layer) multiplied by its height (the total number of layers).



one layer   five layers fill the box

The same layering strategy is used to generalize the method for finding the volume of any prism. The volume of any prism is the area of its base multiplied by its height. The figures above show a base layer that is 3 units by 2 units. However, this strategy will work for base dimensions that are not whole numbers. In that case, you can use smaller unit cubes so that a whole number of cubes covers the base exactly.



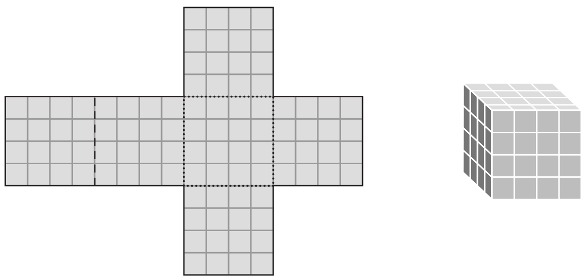
Rectangular Prism   Triangular Prism   Hexagonal Prism

Cylinders

Like a prism, a cylinder has two identical bases (that are circles). Also like a prism, a cylinder has a lateral surface that flattens to a rectangle with height equal to the height of the cylinder and width equal to the circumference of the base. Cylinders can be thought of as circular prisms. In this case, it is natural to extend the techniques for measuring prisms to techniques for measuring cylinders. For more information about the development of student understanding of circles, see the Math Background page **Area and Circumference of a Circle**.

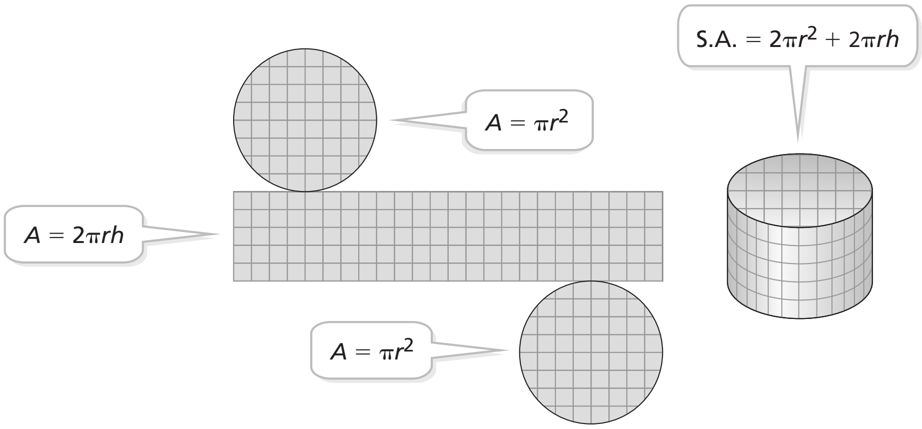
Surface Area of a Cylinder

In earlier grades, students explored the surface area of a rectangular box. Students made nets that can be folded into boxes. The area of the net becomes the surface area of the box. This provides a visual representation of surface area as a two-dimensional attribute, though it is an attribute of a three-dimensional object.



In this Unit, students continue to generalize the surface area of any right prism as the areas of the bases plus the area of the rectangle that forms the lateral faces.

Students cut and fold a net to form a cylinder. In the process, their observations lead to the general formula S.A. $= 2\pi r^2 + 2\pi rh$ (the areas of the circular bases plus the area of the lateral surface that unfolds to a rectangle).



If students need language for the rectangle that is part of the net of both cylinders and prisms, feel free to introduce the term *lateral surface*. Students are familiar with the word *lateral* from their work in Investigation 2.

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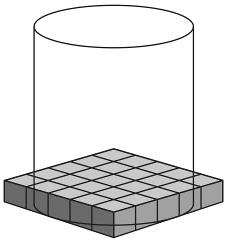
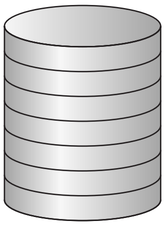
Volume of a Cylinder

The volume of a cylinder is developed as the number of unit cubes in one layer (the area of the circular base) multiplied by the number of layers (the height) needed to fill the cylinder.



Estimate the number of unit cubes in one layer.



Then multiply by the number of layers.

Because the edge of the circular base intersects the unit cubes, no matter how small the chosen units are, students need another strategy besides counting unit cubes to find the area of the base. In Investigation 3 of this Unit, students learn the formula for the area of a circle. They can apply this formula to find the area of the base of a cylinder. The area of the base is multiplied by the height to find the volume. So, the formula for the volume of a cylinder is $V = \pi r^2 h$.

Students also investigate the relationship between right polygonal prisms and cylinders. Suppose several prisms have the same base perimeter and the same height. The lateral area remains constant, but the volume increases as the number of sides of the base increases. A cylinder with the same base perimeter and height has a greater volume (and base area) than any of the prisms.

Area and Circumference of a Circle

The Unit *Covering and Surrounding* highlighted two important kinds of measures (perimeter and area) that depend on different units and measurement processes. Counting is a natural and appropriate way for students to find area and perimeter. When they measure, they are counting the number of measurement units needed to “match” an attribute of an object.

Measuring perimeter requires linear units. Measuring area requires square units. When finding the perimeter of a figure, students will often say they counted the number of squares along a side to find the length. Students need to be aware that perimeter is a linear measure. To measure the perimeter, they count (or measure) the number of units that form the border of the figure.

This method of choosing a linear unit and using it to measure the perimeter works for any figure, including those with curves, such as circles. However, students use a particular unit of length to find the circumference of a circle—the diameter. Students count the number of diameter lengths d needed to surround the circle. The number is about 3. After finding that the ratio of circumference to diameter is approximately 3.14, students develop the formula for the circumference of a circle as $C = \pi d$, or $C = 2\pi r$, where r is the radius.

In *Covering and Surrounding*, students explored areas of polygons and found formulas for rectangles, parallelograms, and triangles. The idea of covering naturally extends to nonpolygonal figures, such as circles. This Unit includes an Investigation from the CMP2 geometry Unit *Covering and Surrounding*, since understanding area and circumference of circles is called for in Grade 7 by CCSSM. Contrasting areas of polygons with the areas of circles provides a deeper understanding of each. Once the formulas for area and circumference of a circle have been developed, it seems natural to explore the volume and surface area of cylinders. This sequence provides an important application of circles and strengthens the understanding of both circles and cylinders.

In Investigation 3, students find the area of a circle by counting the number of unit squares needed to exactly cover the circle. At first, we leave the result as an estimate. However, in the process, some students find interesting ways to count. Some might divide the circle into several congruent sectors, count the squares needed to cover one of the sectors, and then multiply by the number of sectors. Some might enclose the circle inside a square, and then subtract off the pieces of the square that lie outside the circle. Some students might enclose the circle inside a large square and then subtract off the unit squares and parts of unit squares that lie outside the circle. For students who embed the circle in a square of side length d, they will notice that the area is approximately 75%, or $\frac{3}{4}$, of d^2. These methods can lead to the more standard formula for the area of a circle.

The standard formula for the area of a circle is developed by finding the number of squares, whose side-lengths are equal to the radius, that cover the circle. The circle is enclosed in a square. Two perpendicular diameters are drawn, which makes four squares, each with area r^2. The area of the circle is less than $4r^2$. Then by either finding the number of radius squares that cover the circle or by counting the area outside the circle and inside the larger square, the area of the circle is approximately 3 radius squares [$\pi(r \times r)$, or πr^2]. Notice that both methods yield the same result of approximately $3r^2$.

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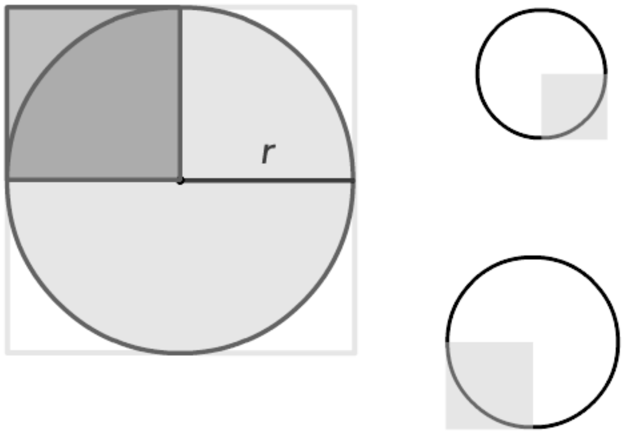
One possible method is shown below. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete animation.

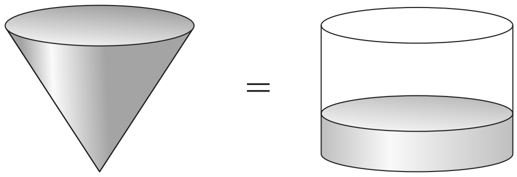


$A = \pi r^2$

Another way to find the area of a circle is to divide the circle into sectors and reassemble the sectors to make a shape that looks roughly like a parallelogram. The more sectors the circle is divided into, the more closely the shape resembles a parallelogram. The length of this parallelogram is half the circumference of the circle, and the width is the radius of the circle. So, the area of the parallelogram, which is equal to the area of the circle, is $\frac{1}{2}Cr$. By substituting in the formula for circumference, we get $\frac{1}{2}\pi dr$, which simplifies to $\frac{1}{2}\pi(2r)r$, or πr^2. The Launch Video in Problem 2.3 illustrates this method.

Spheres, Cones, and Pyramids

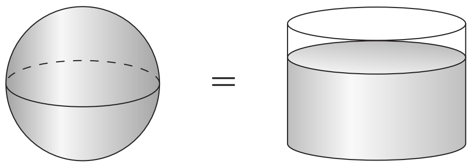
Derivation of the formulas for surface area and volume of spheres, cones, and pyramids is mathematically beyond the scope of this course. However, in Investigation 4, students conduct experiments that suggest what some of the formulas might be. If a cylinder, a cone, and a sphere all have the same radius and the same height (the height being equal to two radii), then it takes three cones full of sand to fill the cylinder and one and a half spheres full of sand to fill the cylinder. **Pouring and Filling** is an activity that allows them to do the same experiment on a computer. Some of the relationships students discover are shown below. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete image gallery.



$$V_{\text{cone}} = \frac{1}{3}(\pi r^2 h)$$

or

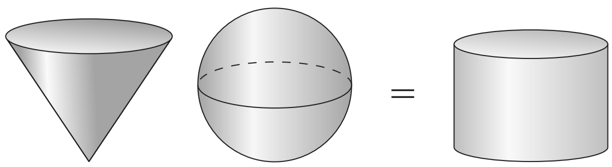
$$V_{\text{cone}} = \frac{2}{3}(\pi r^3)$$



$$V_{\text{sphere}} = \frac{2}{3}(\pi r^2 h)$$

or

$$V_{\text{sphere}} = \frac{4}{3}(\pi r^3)$$



$$V_{\text{cone}} + V_{\text{sphere}} = V_{\text{cylinder}}$$

Students then use these relationships to determine formulas for finding the volumes of any of the three solid figures. In the Grade 8 Unit *Say It With Symbols*, students will further explore these more symbolic relationships between solids.

For cones and spheres, only the volume is studied. Surface area in these two cases is not considered here because the reasoning needed would take us too far from the understanding students currently have about surface area. Formulas for these measurements are sometimes considered within the context of high school geometry or calculus courses. In the ACE for Investigation 4, students attempt to cover a sphere with “radius circles.” Because it takes about 4 such circles, this suggests that the formula for surface area of a sphere is $A = 4\pi r^2$.

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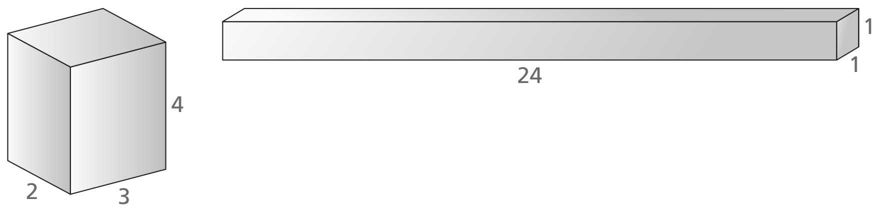
Students find the volume of a square pyramid in a similar way: by comparing it to a square prism. The relationship between the volumes of a square pyramid and its related square prism is analogous to the relationship between the volumes of a cone and its related cylinder. This relationship is explored in the ACE for Investigation 4 and is generalized to finding the volume of any pyramid using the volume of its related prism.

If the base of the pyramid is a polygon, then as the number of sides of the polygon increases, the shape of the pyramid gets closer to a cone. The general formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height.

Relationship Between Surface Area and Volume

In *Covering and Surrounding*, students discovered the surprising fact that plane figures with the same area do not necessarily have the same perimeter and vice versa. For solid figures, there is an analogous "nonresult" that figures with the same volume do not necessarily have the same surface area.

Students investigate the effects of a change in dimension, surface area, or volume on the other attributes of a three-dimensional object. For example, suppose 24 unit cubes are arranged in a rectangular shape and packaged in a rectangular box. There is an arrangement that requires the least packaging material and an arrangement that requires the most packaging material. By physically arranging the cubes and determining the surface area of each arrangement, students discover that a column of 24 cubes requires the most packaging and that the arrangement that is most like a cube (2 by 3 by 4) requires the least amount of packaging.



For rectangular prisms of fixed volume, the shape that has least surface area is a cube. This is similar to ideas students have studied about plane figures: for a fixed area, the rectangle that is most like a square has the least perimeter.

These principles have practical implications when decisions about packaging products are made—the cheapest box design to hold a given volume of material will be in the shape of a cube. Similarly, the cylindrical container with fixed volume and least surface area will have height equal to the diameter of the base (or double the radius of the base).

For solid figures in general, the shape with fixed volume and minimum surface area is not a cube but a sphere. For a fixed volume, a sphere has the least surface area and, conversely, for a fixed surface area, a sphere has the largest volume.

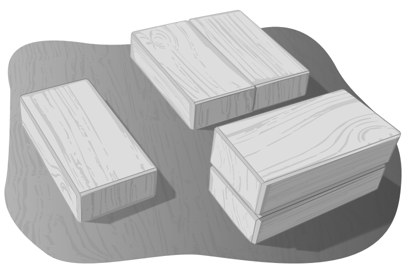
Scaling Solid Figures

In *Stretching and Shrinking*, students discovered that when a planar figure is stretched (or shrunk) to a similar shape by a transformation with scale factor k, all lengths in the transformed figure are k times their corresponding lengths in the original figure, and all areas in the transformed figure are k^2 times their corresponding areas in the original figure. The same principle applies to solid figures as well.

In this Unit, students also look at the effects that doubling all three dimensions of a box have on the volume and surface area of the box.

Making scale models of the original box and the new box helps students visualize the effects of the scale factor. Doubling each dimension of a rectangular prism increases the surface area by $2 \times 2 = 4$ times (a scale factor of 2^2) and volume by $2 \times 2 \times 2 = 8$ times (a scale factor of 2^3). The surface areas of the two prisms, if viewed as nets, are similar figures with a scale factor of 2 from the net of the small prism to the net of the large prism.

A common misconception is confronted when students are asked to investigate what size box would have twice the volume of a box of given dimensions. They find that they need to double only one dimension of a rectangular box to double its volume.



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When a solid figure is stretched or shrunk to a similar shape by transformation with scale factor k, all lengths in the transformed figure are k times their corresponding lengths in the original. The surface area of the transformed figure is k^2 times the area of the original figure, and the volume of the transformed figure is k^3 times that of the original figure. This principle can be discovered in the case of rectangular prisms using experimentation with different values of k and confirmed algebraically using the surface area and volume formulas for those figures.

$S.A._0 = 2(\ell w + \ell h + wh)$	Write the formulas for surface area and volume.	$V_0 = \ell wh$
$S.A._1 = 2[(k\ell)(kw) + (k\ell)(kh) + (kw)(kh)]$	Multiply each dimension by k.	$V_1 = (k\ell)(kw)(kh)$
$= k^2[2(\ell w + \ell h + wh)]$	Factor out a power of k.	$= k^3(\ell wh)$
$= k^2\, S.A._0$	Substitute $S.A._0$ or V_0.	$= k^3\, V_0$

When we describe a cylinder, we generally give only two dimensions: the height and the radius. However, when we find the area of the base, we use one of these dimensions twice: $\pi \times r \times r$. The way that the standard formula is developed involves an image of a square with area r^2 and we need π of these to cover the base. So, when we scale the radius, we scale both sides of this radius square. Students explore scaling with cylinders and spheres and determine that the same relationships apply for surface area (k^2) and volume (k^3).

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